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Kinetic Models for Granular Flow

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The generalization of the Boltzmann and Enskog kinetic equations to allow inelastic collisions provides a basis for studies of granular media at a fundamental level. For elastic collisions the significant technical challenges presented in solving these equations have been circumvented by the use of corresponding model kinetic equations. The objective here is to discuss the formulation of model kinetic equations for the case of inelastic collisions. To illustrate the qualitative changes resulting from inelastic collisions the dynamics of a heavy particle in a gas of much lighter particles is considered first. The Boltzmann–Lorentz equation is reduced to a Fokker–Planck equation and its exact solution is obtained. Qualitative differences from the elastic case arise primarily from the cooling of the surrounding gas. The excitations, or physical spectrum, are no longer determined simply from the Fokker–Planck operator, but rather from a related operator incorporating the cooling effects. Nevertheless, it is shown that a diffusion mode dominates for long times just as in the elastic case. From the spectral analysis of the Fokker–Planck equation an associated kinetic model is obtained. In appropriate dimensionless variables it has the same form as the BGK kinetic model for elastic collisions, known to be an accurate representation of the Fokker–Planck equation. On the basis of these considerations, a kinetic model for the Boltzmann equation is derived. The exact solution for states near the homogeneous cooling state is obtained and the transport properties are discussed, including the relaxation toward hydrodynamics. As a second application of this model, it is shown that the exact solution for uniform shear flow arbitrarily far from equilibrium can be obtained from the corresponding known solution for elastic collisions. Finally, the kinetic model for the dense fluid Enskog equation is described.

KEY WORDS: Granular flow; kinetic theory; Fokker–Planck equation; Boltzmann equation; Enskog equation; kinetic model.

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I. INTRODUCTION

The primary feature of granular media differentiating them from simple atomic fluids is inelastic collisions.$^{(1)}$ Standard kinetic equations (e.g., Boltzmann and Enskog equations) can be adapted to incorporate this feature and provide the theoretical basis for studying the dynamical properties of granular media.$^{(2-4)}$ However, the complexity of these non-linear integro-differential equations in the six dimensional phase space places severe limitations on applications just as for elastic collisions. In the latter case, kinetic models have played an important role in providing practical access to information about nonequilibrium states that would not be available otherwise.$^{(5)}$ The spirit of a kinetic model is to retain essential features of the underlying kinetic equation such as macroscopic balance equations and stationary states, while otherwise introducing simplifications that admit more detailed analysis. In constructing a kinetic model there is always some subjective compromise between quantitative accuracy and simplicity, usually dictated by the objectives of the application. Kinetic models originate from approximations to the collision operator. For elastic collisions they typically preserve the invariance properties associated with conservation laws, but otherwise have a single characteristic frequency supplanting the remaining complex spectrum for microscopic excitations. Although they are crude in this latter respect, they contain the essential ingredients of a collision operator generating two widely separated time scales: that for hydrodynamics and that for kinetics.

The introduction of kinetic models for inelastic collisions proceeds in much the same way.$^{(3,6-8)}$ Although energy is no longer conserved, the kinetic model is nevertheless constrained to yield the corresponding macroscopic balance equation. Similarly, the spectrum for kinetic excitations is collapsed to a single characteristic feature. An important difference in the underlying kinetic equation, however, is that no homogeneous stationary state exists. Instead, there is a homogeneous cooling state (HCS) such that the thermal velocity is monotonically decreasing in time. The inverse cooling rate introduces a new time scale not present for elastic collisions and requires some special care in constructing the kinetic model. The objective here is to analyze such questions in detail and to suggest simple kinetic models appropriate for the study of granular flow in complex states.

The most important effects of inelastic collisions are illustrated in the next two sections for the tagged particle dynamics of a heavy particle in a gas of much lighter particles. In this case, the Boltzmann–Lorentz kinetic equation can be reduced exactly to a Fokker–Planck equation with drift and diffusion coefficients depending on the time dependent temperature of the surrounding fluid in its HCS and on the coefficient of normal restitution for collisions between the Brownian particle and fluid particles. The spectrum of the collision operator is known and the exact solution can be constructed to demonstrate the qualitative changes due to inelastic collisions and for tests of appropriate kinetic models. For example, it is found that the usual fluctuation-dissipation relation or detailed balance condition does not hold for inelastic collisions. As a consequence, the final quasi-stationary state approached for long times is a Maxwellian at a different temperature from that of the surrounding gas due to the inability of the tagged particle to adjust exactly to the cooling of the gas. Nevertheless, in appropriate reduced variables accounting for this time dependent cooling, the spectrum of the Fokker–Planck equation is similar to that for elastic collisions. There are an infinite number of discrete kinetic modes and a single "hydrodynamic" diffusive mode. All of the kinetic modes are shifted by a small amount due to the cooling rate. This small quantitative effect on the spectrum is due to the cooling and requires special attention when constructing a kinetic model based on simplification of the spectrum, as discussed in Section III. The hydrodynamic pole is always smaller than those for the kinetic modes so for asymptotically long times the dynamics is dominated by simple diffusion. This is a clear demonstration of the existence of hydrodynamics in a system with inelastic collisions, for all values of the restitution coefficient, and gives a precise description of the approach to the hydrodynamic stage for arbitrary initial conditions. In real time, however, the separation of hydrodynamic and kinetic modes is algebraic rather than exponential.

With the spectrum of the Fokker–Planck equation known, it is possible to test the application of kinetic models for approximating kinetic equations. In the simplest model for the case of elastic collisions, the entire spectrum for the kinetic modes is collapsed to a single infinitely degenerate pole located at the average collision frequency, leading to the BGK kinetic model.(5) In Section III it is shown that this approach applies as well to the case of inelastic collisions, but only after the time dependence of the fluid temperature has been removed by appropriate scaling. In dimensionless units the BGK kinetic model is regained with only a renormalization of the collision frequency and a non-linear transformation of the time scale. As this model is known to be a good approximation for elastic collisions, we confirm its validity for the case of inelastic collisions as well. In particular, the detailed description of "aging to hydrodynamics" is the same as for the elastic case, with the same condition that the wavevector not exceed a critical value. In this way, it is verified that the approach to hydrodynamics for all values of the restitution coefficient is preserved by the kinetic model.

In Section IV these results are used to motivate and guide construction of a kinetic model for the Boltzmann equation for a low density gas with inelastic collisions. The result extends and clarifies previous studies$^{(3, 6–8)}$ by a more detailed attention to the choice of representation for the collision operator whose spectrum is being approximated. The model kinetic equation yields the same balance equations for mass, energy, and momentum as the Boltzmann equation, and it supports the same HCS solution as the formal asymptotic state for arbitrary homogeneous initial conditions. The HCS distribution and a constant collision frequency must be supplied as input for the model. The HCS distribution is known to good approximation from the Boltzmann equation using a Sonine polynomial expansion about a Maxwellian, and the collision frequency can be chosen to yield the Boltzmann shear viscosity which also has been calculated recently from a similar expansion.(8) To illustrate its application, the solution to the corresponding linearized kinetic model equation is obtained for initial conditions close to the HCS. The spectrum consists of damped single particle excitations and branch cuts corresponding to kinetic modes that decay on the time scale of the inverse collision frequency. In addition, there are $d+2$ hydrodynamic modes (d is the dimension of the system) just as for a gas with elastic collisions which dominate for long times and large wavelengths. Again, the separation of hydrodynamic and kinetic modes is algebraic rather than exponential in time. The approach to hydrodynamics is described in more detail for transverse excitations where it is possible to map the inelastic case onto known exact results for the elastic one. This result shows the relaxation of fast modes to slow hydrodynamics including wavelengths shorter than those required for the Navier–Stokes limit, and for arbitrary value of the restitution coefficient. As a second illustration the non-linear kinetic model is applied to uniform shear flow arbitrarily far from equilibrium. It is observed that the equation for the stationary solution has the same form as that for elastic fluids in the presence of a thermostat, with the dissipation due to inelastic collisions playing the same role as the thermostat. In the latter case, the effect of the inelastic collisions for fixed restitution coefficient is controlled by the temperature which adjusts autonomously to yield a stationary state with viscous heating and dissipation balanced. This correspondence allows direct translation of extensive previous studies for non-dissipative systems to granular flows. The exact distribution function, shear rate dependence of the temperature, and viscosity are given as examples.

Finally, the generalization of this kinetic model to one for the dense fluid revised Enskog kinetic equation(9) is given in Section V. The form follows from the considerations of Section IV and the recent formulation of an Enskog kinetic model for the elastic case.(10, 11) The primary results of this work and the implications to be drawn from them are summarized in the last section.

II. TAGGED PARTICLE MOTION

In this section, some of the most important new features of kinetic theory for systems of particles colliding inelastically are illustrated for the special case of a heavy tagged particle in a gas of lighter particles. The Boltzmann–Lorentz kinetic equation is expressed as a Fokker–Planck equation in the limit of small mass ratio, and its exact spectrum is described. Several qualitative differences from the case of elastic collisions are noted. These results will be used in the next section to guide the construction of an appropriate kinetic model.

We consider a tagged particle of mass m in a low density gas whose particles have mass m_g. All particles are smooth hard spheres ($d=3$) or disks ($d=2$). Collisions between the tagged particle and fluid particles are characterized by a constant coefficient of normal restitution α, while that for collisions among fluid particles will be denoted by α_g. Both coefficients have values $0<\alpha, \alpha_g \leqslant 1$, with the largest value corresponding to the elastic limit.

The Boltzmann–Lorentz equation for the probability density $F(\mathbf{r}, \mathbf{v}, t)$ of the tagged particle is

$$(\partial_t + \mathbf{v} \cdot \nabla) F = J[\mathbf{r}, \mathbf{v}, t \mid F, f] \tag{2.1}$$

where the collision operator J is given by refs. 2–4

$$J[\mathbf{r}, \mathbf{v}, t \mid F, f] = \sigma_0^{d-1} \int d\mathbf{v}_1 \int d\hat{\boldsymbol{\sigma}}\, \Theta(\mathbf{g} \cdot \hat{\boldsymbol{\sigma}})(\mathbf{g} \cdot \hat{\boldsymbol{\sigma}})$$
$$\times [\alpha^{-2} F(\mathbf{r}, \mathbf{v}', t)\, f(\mathbf{r}, \mathbf{v}_1', t) - F(\mathbf{r}, \mathbf{v}, t)\, f(\mathbf{r}, \mathbf{v}_1, t)] \tag{2.2}$$

Here $f(\mathbf{r}, \mathbf{v}, t)$ is the one-particle distribution for the surrounding gas, Θ is the Heaviside step function, $\hat{\boldsymbol{\sigma}}$ is a unit vector pointing from the center of gas particle 1 to the center of the tagged particle at contact, and $\sigma_0 = (\sigma + \sigma_g)/2$, where σ and σ_g are the diameters of the tagged particle and the gas particles, respectively. The precollisional or restituting velocities $\mathbf{v}'$ and $\mathbf{v}_1'$ are given by

$$\mathbf{v}' = \mathbf{v} - \frac{(1+\alpha)\Delta}{\alpha(1+\Delta)} (\mathbf{g} \cdot \hat{\boldsymbol{\sigma}})\, \hat{\boldsymbol{\sigma}}, \qquad \mathbf{v}_1' = \mathbf{v}_1 + \frac{(1+\alpha)}{\alpha(1+\Delta)} (\mathbf{g} \cdot \hat{\boldsymbol{\sigma}})\, \hat{\boldsymbol{\sigma}} \tag{2.3}$$

In the above expressions $\mathbf{g} = \mathbf{v} - \mathbf{v}_1$ and Δ is the mass ratio of a gas particle to the tagged particle, $\Delta = m_g/m$.

The gas distribution $f(\mathbf{r}, \mathbf{v}, t)$ must be a solution of the Boltzmann equation for inelastic particles. Here we will consider that the gas is in the HCS that is described by a solution of the form$^{(2)}$:

$$f_H(\mathbf{v}, t) = n_g v_g^{-d}(t)\, \phi\left(\frac{v}{v_g(t)}\right) \tag{2.4}$$

where $v_g(t) = [2k_B T_g(t)/m_g]^{1/2}$ is the thermal velocity of the gas particles. The number density n_g and the temperature of the gas $T_g(t)$ are defined in the usual way,

$$n_g = \int d\mathbf{v}\, f_H(\mathbf{v}, t), \qquad \frac{d}{2} n_g k_B T_g(t) = \int d\mathbf{v}\, \frac{1}{2} m_g v^2 f_H(\mathbf{v}, t) \tag{2.5}$$

As indicated by Eq. (2.4), all the time dependence of the distribution f_H takes place through the temperature of the gas. An evolution equation for it is easily obtained from the Boltzmann equation,

$$\frac{dT_g}{dt} = -\zeta(T_g)\, T_g \tag{2.6}$$

where the cooling rate $\zeta(T_g)$ is given by

$$\zeta(T_g) = (1 - \alpha_g^2)\, n_g \sigma_g^{d-1} \left(\frac{k_B T_g}{m_g}\right)^{1/2} \tilde{\omega} \tag{2.7}$$

with $\tilde{\omega}$ being a numerical dimensionless factor,

$$\tilde{\omega} = \frac{\pi^{(d-1)/2}}{2^{1/2} d\Gamma((d+3)/2)} \int d\mathbf{v} \int d\mathbf{v}_1\, g^3 \phi(v)\, \phi(v_1) \tag{2.8}$$

The solution of Eq. (2.6) is

$$T_g(t) = T_g(0)\left(1 + \frac{\zeta[T_g(0)]}{2}\, t\right)^{-2} \tag{2.9}$$

The above applies for arbitrary masses of the tagged and gas particles. For the special case of a massive tagged particle in a freely evolving inelastic dilute gas, the Boltzmann–Lorentz equation simplifies to a

Fokker–Planck equation, following a formal expansion to leading order in the mass ratio Δ for $\Delta \to 0$:

$$(\partial_t + \mathbf{v} \cdot \nabla)\, F(\mathbf{r}, \mathbf{v}, t) = \mathscr{L}[T_g(t)]\, F(\mathbf{r}, \mathbf{v}, t) \tag{2.10}$$

$$\mathscr{L}(T_g) = \gamma_e(T_g)\, a_0(\alpha) \frac{\partial}{\partial \mathbf{v}} \cdot \left[\mathbf{v} + \frac{k_{\mathrm{B}} T_g}{m} a_0(\alpha) \frac{\partial}{\partial \mathbf{v}} \right] \tag{2.11}$$

where $\gamma_e(T_g)$ is the friction constant for the elastic limit, except as a function of the cooling temperature,

$$\gamma_e(T_g) = \frac{4\pi^{(d-1)/2} \sigma_0^{d-1} n_g \Delta^{1/2}}{d\Gamma(d/2)} \left(\frac{2k_{\mathrm{B}} T_g}{m} \right)^{1/2} \tag{2.12}$$

All effects of inelastic collisions among the gas particles appear through the time dependence of the gas temperature given by Eq. (2.6). The inelasticity of collisions between the tagged particle and gas particles shows up through the function

$$a_0(\alpha) = \frac{1 + \alpha}{2} \tag{2.13}$$

For the case of elastic collisions, the coefficient of the second derivative in Eq. (2.11) is equal to $k_{\mathrm{B}} T_g / m$ times that of the first derivative, a fluctuation-dissipation relation. In the present case, this relation is violated by the additional factor of $a_0(\alpha)$. The derivation of Eq. (2.10) follows from an extension of the standard method for elastic particles$^{(12, 13)}$ and is outlined in Appendix A. An important difference for the inelastic case is the requirement of two limits, $\Delta \to 0$ and $\alpha_g \to 1$, with

$$\varepsilon_0 \equiv \frac{\zeta[T_g(t)]}{2a_0(\alpha)\, \gamma_e[T_g(t)]} = \frac{1 - \alpha_g^2}{4\sqrt{2}\, a_0(\alpha)\, \Delta} \left(\frac{\sigma_g}{\sigma_0} \right)^{d-1} \to \text{constant} < 1 \tag{2.14}$$

Thus, in addition to small mass ratio the gas dissipation must be weak (there are no limitations on α for the tagged particle, however). To understand this, define a tagged particle temperature $T(t)$ from its average kinetic energy by

$$\frac{d}{2} k_{\mathrm{B}} T = \int d\mathbf{r} \int d\mathbf{v}\, \frac{1}{2} m(\mathbf{v} - \mathbf{u})^2\, F \tag{2.15}$$

where

$$\mathbf{u}(t) = \int d\mathbf{r} \int d\mathbf{v}\; \mathbf{v} F \tag{2.16}$$

is the average velocity of the tagged particle. Then, taking the average of the Fokker–Planck equation with respect to $\mathbf{v}$ and v^2 gives

$$\partial_t \mathbf{u} = -\gamma_e(T_g)\, a_0(\alpha)\, \mathbf{u} \tag{2.17}$$

$$[\partial_t + 2\gamma_e(T_g)\, a_0(\alpha)(1-\varepsilon_0)]\, \frac{T(t)}{T_g(t)} = 2\gamma_e(T_g)\, a_0^2(\alpha) \tag{2.18}$$

The first of these equations shows that $\gamma_e a_0$ is the decay rate for the average velocity of the tagged particle due to collisions with the gas. The second velocity moment, or $T(t)$, has a decay rate of $2\gamma_e a_0$. This provides the interpretation of ε_0 in Eq. (2.14) as the ratio of cooling rate for the gas relative to that for the tagged particle. Consequently, $-2\gamma_e a_0(1-\varepsilon_0)$ in Eq. (2.18) represents the difference between these two decay rates, and expresses the "frustrated" thermalization of the tagged particle due to the cooling of the surrounding gas. The solutions of Eqs. (2.17) and (2.18) have the forms

$$\mathbf{u}(t) = [\, U(t)]^{1/2(1-\varepsilon_0)}\, \mathbf{u}(0) \tag{2.19}$$

$$\frac{T(t)}{T_g(t)} = U(t)\, \frac{T(0)}{T_g(0)} + a_0(\alpha)(1-\varepsilon_0)^{-1}\, [1 - U(t)] \tag{2.20}$$

where

$$U(t) = \left(1 + \frac{\zeta[T_g(0)]}{2}\, t\right)^{-2(1-\varepsilon_0)/\varepsilon_0} \tag{2.21}$$

This is similar to the algebraic decay of $T_g(t)$ given in Eq. (2.9) except with an exponent reflecting the differences in the decay of the two temperatures. The coefficients in the Δ expansion depend on the ratio of temperatures for the tagged and gas particles (see Appendix A). For $\varepsilon_0 \geqslant 1$, $T(t)/T_g(t)$ grows without bound while for $\varepsilon_0 < 1$ it approaches the constant value

$$\lim_{t\to\infty} \frac{T(t)}{T_g(t)} = a_0(\alpha)(1-\varepsilon_0)^{-1} \tag{2.22}$$

This is the origin of condition (2.14).

Although Eq. (2.22) implies that the cooling rates for the gas and tagged particles approach the same value, the respective asymptotic temperatures can be different. Thus for the choice $(1-\alpha)/2 < \varepsilon_0 < 1$ the asymptotic temperature of the tagged particle is greater than that for the surrounding gas, while for $0 < \varepsilon_0 < (1-\alpha)/2$ the tagged particle temperature is less than that for the gas. Also, the approach to the asymptotic value is algebraic rather than exponential in time,

$$\begin{aligned} T(t) &\to a_0(\alpha)(1-\varepsilon_0)^{-1}\, T_g(t) \\ &= a_0(\alpha)(1-\varepsilon_0)^{-1}\, T_g(0)\left(1+\frac{\zeta[T_g(0)]}{2}\,t\right)^{-2} \end{aligned} \tag{2.23}$$

The asymptotic difference between T and T_g arises from two competing effects. On the one hand, the inelasticity of the collisions between the tagged and gas particles tends to decrease the tagged particle temperature. Thus, even if the gas particles are elastic ($\varepsilon_0 = 0$), $T/T_g \to a_0(\alpha) < 1$. On the other hand, the cooling of the gas particles tends to make T lag behind T_g, despite the fact that the relaxation rate of T, $2\gamma_e(T_g)\, a_0(\alpha)$, is assumed to be larger than the cooling rate of the gas particles, $\zeta(T_g)$. Consequently, even if the tagged particle collides elastically with the gas particles ($a_0 = 1$), $T/T_g \to (1-\varepsilon_0)^{-1} > 1$. Both competing effects exactly balance each other only if $\varepsilon_0 = (1-\alpha)/2$.

In the absence of external forces or boundary conditions there is no stationary solution to the Fokker–Planck equation, due to the explicit time dependence of the gas temperature $T_g(t)$. However, there is a HCS analogous to that for the gas, Eq. (2.4),

$$F_H(\mathbf{v}, t) = \Omega^{-1} v_0^{-d}(t)\, \Phi(v/v_0(t)) \tag{2.24}$$

where Ω is the volume of the system and $v_0(t) = [2k_{\mathrm{B}} T(t)/m]^{1/2}$ is the thermal velocity for the tagged particle. Substitution of Eq. (2.24) into Eq. (2.10) gives

$$-\frac{1}{2T(t)} \frac{\partial T(t)}{\partial t} \frac{\partial}{\partial \mathbf{v}} \cdot [\mathbf{v} F_H(\mathbf{v}, t)] = \mathscr{L}[T_g(t)]\, F_H(\mathbf{v}, t) \tag{2.25}$$

Using Eq. (2.18) this can be rearranged as

$$\frac{\partial}{\partial \mathbf{v}} \cdot \left[\mathbf{v} + \frac{k_{\mathrm{B}} T(t)}{m} \frac{\partial}{\partial \mathbf{v}}\right] F_H(\mathbf{v}, t) = 0 \tag{2.26}$$

with the solution

$$F_H(\mathbf{v}, t) = \Omega^{-1}\, \frac{v_0^{-d}(t)}{\pi^{d/2}}\, e^{-v^2/v_0^2(t)} \tag{2.27}$$

The HCS for the tagged particle is a Maxwellian, just as in the case of elastic collisions, but as a function of the time dependent tagged particle temperature. The importance of this special solution lies in the fact that the general solution approaches $F_H(\mathbf{v}, t)$ for large times. Consider first the case of homogeneous initial conditions. The general solution to the Fokker–Planck equation can be obtained in the same way as that for elastic collisions$^{(13, 14)}$:

$$F(\mathbf{v}, t) = \int d\mathbf{v}'\, G(\mathbf{v}, t; \mathbf{v}', 0)\, F(\mathbf{v}', 0) \tag{2.28}$$

where

$$G(\mathbf{v}, t; \mathbf{v}', 0) = \left\{ \frac{(1-\varepsilon_0)\, m}{2\pi k_{\mathbf{B}} T_g(t)\, a_0(\alpha)[1-U(t)]} \right\}^{d/2} \times \exp\left\{ -\frac{(1-\varepsilon_0)\, m[\mathbf{v} - w(t)\, \mathbf{v}']^2}{2k_{\mathbf{B}} T_g(t)\, a_0(\alpha)[1-U(t)]} \right\} \tag{2.29}$$

Here $w(t) \equiv [U(t)]^{1/2(1-\varepsilon_0)}$ governs the decay of the average velocity of the tagged particle, cf. Eq. (2.19). This has a form similar to that for the elastic case, although here the relevant "temperature" is $T_g(t)\, a_0(\alpha)(1-\varepsilon_0)^{-1}$. However, from Eq. (2.20) it follows that this quantity agrees with the actual temperature of the tagged particle in the limit of asymptotically long times. Qualitatively, an initially sharp distribution peaked about $\mathbf{v}'$ shifts towards the origin on a time scale of the velocity relaxation, $\{a_0(\alpha)\, \gamma_e[T_g(t)]\}^{-1}$. In addition, the width of the distribution broadens (thermalization of the particle) on the time scale $\{2(1-\varepsilon_0)\, a_0(\alpha) \times \gamma_e[T_g(t)]\}^{-1}$. As the tagged particle relaxes toward the gas temperature the latter changes, leading to a slowing of the process by the factor $1-\varepsilon_0$. As indicated in (2.21) the functional dependence for both processes is algebraic in time, contrasting the exponential decay for the case of elastic collisions.

It is instructive for the purpose of kinetic modeling in the next section to repeat the discussion of velocity relaxation from a different perspective by analyzing the spectrum of the Fokker–Planck operator. The operator $\mathscr{L}$ defined in Eq. (2.11) has a discrete spectrum with eigenvalues labelled by d positive or null integers $\mathbf{n} = (n_1, ..., n_d)$:$^{(13, 15)}$

$$\lambda_{\mathbf{n}}[T_g(t)] = -a_0(\alpha)\, \gamma_e[T_g(t)] \sum_{i=1}^{d} n_i \tag{2.30}$$

For an elastic gas ($\alpha_g = 1$) the eigenvalues are time independent and they represent the possible decay rates for the tagged particle distribution function. On the other hand, for inelastic collisions the relationship of the eigenvalues to the modes of $F(\mathbf{v}, t)$ is not so direct due to the additional time dependence of the eigenvalues and eigenfunctions through the temperature of the bath. This difficulty can be removed by transforming the Fokker–Planck equation to one with constant coefficients using the dimensionless velocity, distribution function, and time,

$$\mathbf{v}^* = \frac{\mathbf{v}}{\hat{v}_0(t)}, \qquad F^*(\mathbf{v}^*, t^*) = \Omega \hat{v}_0^d(t)\, F(\mathbf{v}, t), \qquad t^*(t) = a_0(\alpha) \int_0^t dt'\, \gamma_e[T_g(t')] \tag{2.31}$$

where $\hat{v}_0(t) \equiv [2k_B T_g(t)\, a_0(\alpha)/m(1-\varepsilon_0)]^{1/2}$. Then Eq. (2.10) for homogeneous states becomes a differential equation with time independent coefficients,

$$\partial_{t^*} F^* = -\varepsilon_0 \frac{\partial}{\partial \mathbf{v}^*} \cdot (\mathbf{v}^* F^*) + \mathscr{L}^* F^* \tag{2.32}$$

where

$$\mathscr{L}^* = \frac{\partial}{\partial \mathbf{v}^*} \cdot \left(\mathbf{v}^* + \frac{1-\varepsilon_0}{2} \frac{\partial}{\partial \mathbf{v}^*} \right) \tag{2.33}$$

The Fokker–Planck operator $\mathscr{L}^*$ has the same spectrum as $\mathscr{L}$ in the appropriate dimensionless form:

$$\lambda_{\mathbf{n}}^* = -\sum_{i=1}^{d} n_i \tag{2.34}$$

However, the scaling of the velocity and distribution function to obtain constant coefficients has generated a new velocity derivative operator on the right side of Eq. (2.32) which has a positive spectrum. The full spectrum of the combined operators is easily identified by rewriting them as

$$\mathscr{L}^{*\prime} \equiv -\varepsilon_0 \frac{\partial}{\partial \mathbf{v}^*} \cdot \mathbf{v}^* + \mathscr{L}^* = (1-\varepsilon_0) \frac{\partial}{\partial \mathbf{v}^*} \cdot \left(\mathbf{v}^* + \frac{1}{2} \frac{\partial}{\partial \mathbf{v}^*} \right) \tag{2.35}$$

This is again a Fokker–Planck operator whose spectrum is given by

$$\lambda_{\mathbf{n}}^{*\prime} = -(1-\varepsilon_0) \sum_{i=1}^{d} n_i \tag{2.36}$$

This spectrum is the same as that of Eq. (2.34) except reduced by the factor $(1-\varepsilon_0)$. The eigenfunctions and eigenvalues obtained in these reduced variables are time independent and, therefore, the values $\lambda_{\mathbf{n}}^{*\prime}$ are precisely the decay rates of $F^*(\mathbf{v}^*, t^*)$. Since $\varepsilon_0 < 1$ all excitations are damped. This analysis shows that the effect of the time dependence of the gas temperature is to shift these decay rates by $(1-\varepsilon_0)$ relative to those of the Fokker–Planck operator alone. The transformation to a time independent form of the equation for this identification of the modes generates an additional differential operator, the first term on the right side of (2.32), with large positive eigenvalues. These are controlled and dominated by corresponding large negative eigenvalues of $\mathscr{L}^*$ with the net result that $\mathscr{L}^{*\prime}$ has a negative spectrum, with a slowing of all modes by a factor $(1-\varepsilon_0)$, but still maintaining relaxation towards the HCS. This observation is critical for constructing kinetic models, as discussed in the next section below, indicating that $\mathscr{L}^{*\prime}$ rather than $\mathscr{L}^*$ is the physically relevant operator.

The eigenfunctions of the operator given in Eq. (2.35) corresponding to the eigenvalues in Eq. (2.36) are a Gaussian times Hermite polynomials. Denoting them by $\Psi^*_{\mathbf{n}}(\mathbf{v}^*)$, the general solution to (2.32) can be expressed as

$$F^*(\mathbf{v}^*, t^*) = \sum_{\mathbf{n}} C_{\mathbf{n}} \Psi^*_{\mathbf{n}}(\mathbf{v}^*)\, e^{\lambda^*_{\mathbf{n}} t^*} \tag{2.37}$$

The coefficients $C_{\mathbf{n}}$ are determined by the initial condition $F^*(\mathbf{v}, 0)$. It is straightforward to recover the form in Eqs. (2.28) and (2.29) from this representation.

The general solution to Eq. (2.10) for inhomogeneous states can be expressed in a form to Eq. (2.28) with G being now a Gaussian in both space and velocity variables. The explicit expression integral is quite complex and not particularly instructive for the purposes here. However, the analysis of the spectrum parallels closely that above for the homogeneous case if a Fourier transform in space is performed, so that Eq. (2.10) becomes

$$(\partial_t - i\mathbf{k}\cdot\mathbf{v})\, \tilde{F}(\mathbf{k}, \mathbf{v}, t) = \mathscr{L}[T_g(t)]\, \tilde{F}(\mathbf{k}, \mathbf{v}, t) \tag{2.38}$$

with

$$\tilde{F}(\mathbf{k}, \mathbf{v}, t) = \int d\mathbf{r}\, e^{i\mathbf{k}\cdot\mathbf{r}} F(\mathbf{r}, \mathbf{v}, t) \tag{2.39}$$

The operator $\mathscr{L} + i\mathbf{k}\cdot\mathbf{v}$ has the same spectrum as $\mathscr{L}$ shifted by a constant,(13, 15)

$$\lambda_{\mathbf{n}} = -a_0(\alpha)\, \gamma_e[T_g(t)] \sum_{i=1}^{d} n_i - D_e[T_g(t)]\, k^2 \tag{2.40}$$

where

$$D_e(T_g) \equiv \frac{k_{\mathrm{B}} T_g}{m\gamma_e(T_g)} \tag{2.41}$$

is the diffusion coefficient for the case of elastic collisions. In the elastic case, the eigenvalues are time independent so the distribution function has a diffusive mode $-D_e(T_g)\, k^2$ for $\mathbf{n}=0$, and an infinite set of modes that decay at a rate faster by the amount $a_0(\alpha)\, \gamma_e(T_g) \sum_i n_i$. Since $D_e(T_g)\, k^2 \ll a_0(\alpha)\, \gamma_e(T_g) \sum_i n_i$ for sufficiently small k (long wavelengths), the diffusive mode dominates at sufficiently large times. This is the expected approach to the homogeneous solution, first by velocity relaxation to a hydrodynamic stage and then by spatial diffusion to a homogeneous state. However, as discussed for the homogeneous case above, these eigenvalues are not the relaxation rates for $\tilde{F}(\mathbf{k}, \mathbf{v}, t)$ when there are inelastic collisions since T_g becomes time dependent. The same analysis in terms of dimensionless variables shows that these relaxation rates are given by the spectrum of $\mathscr{L}^{*\prime} + i\mathbf{k}^* \cdot \mathbf{v}^*$,

$$\lambda_{\mathbf{n}}^{*\prime} = -(1-\varepsilon_0) \sum_{i=1}^{d} n_i - \frac{D_e^*}{(1-\varepsilon_0)^2}\, k^{*2} \tag{2.42}$$

where D_e^* and $\mathbf{k}^*$ are the dimensionless forms of D_e and $\mathbf{k}$,

$$D_e^* = D_e\, \frac{a_0(\alpha)\, \gamma_e[T_g]}{\hat{v}_0^2(t)} = \frac{1}{2}\,(1-\varepsilon_0), \qquad \mathbf{k}^* = \mathbf{k}\, \frac{\hat{v}_0(t)}{a_0(\alpha)\, \gamma_e(T_g)} \tag{2.43}$$

Therefore, the distribution function has both kinetic and diffusive modes, with the kinetic modes slowed by the factor $(1-\varepsilon_0)$ as in the homogeneous case and the diffusion mode enhanced by the factor $(1-\varepsilon_0)^{-2}$. Since the eigenvalue for the diffusion mode is the smallest one, it will dominate for sufficiently long times. Then, for example, the probability density

$$n(\mathbf{r}, t) = \int d\mathbf{v}\, F(\mathbf{r}, \mathbf{v}, t) \tag{2.44}$$

for the tagged particle obeys the diffusion equation on this time scale,

$$\left\{\partial_t - \frac{D_e[T_g(t)]}{(1-\varepsilon_0)^2}\, \nabla^2\right\} n(\mathbf{r}, t) = 0 \tag{2.45}$$

The diffusion coefficient can be expressed in terms of the tagged particle temperature for long times using Eq. (2.22)

$$\frac{D_e[T_g(t)]}{(1-\varepsilon_0)^2} = \frac{k_B T}{m\gamma_{\text{eff}}[T_g(t)]} \tag{2.46}$$

The right side has the form of the elastic case, except with an effective friction coefficient, $\gamma_{\text{eff}}[T_g(t)] = (1-\varepsilon_0)\, a_0(\alpha)\, \gamma_e[T_g(t)]$, which characterizes the rate of change of the particle's mean velocity relative to the thermal velocity of the gas.

The discussion of the last paragraph provides a concrete example for the justification of hydrodynamic equations for granular materials. Although there is no asymptotic stationary state, an initial preparation eventually relaxes via the fast kinetic modes to a dominant slow diffusive mode. The diffusive mode relaxes on the longer time scale to a homogeneous state which is time dependent and characterized by the cooling temperature. The cooling of the system during the kinetic and diffusive relaxation changes these processes from exponential to algebraic in time. Nevertheless, the concept of "aging to hydrodynamics" still applies just as for fluids with elastic collisions.

III. KINETIC MODEL FOR THE FOKKER–PLANCK EQUATION

The above exact analysis of the Fokker–Planck equation provides both guidance and a good testing ground for the construction of kinetic models. A kinetic model for the Fokker–Planck equation with *elastic collisions* is constructed by replacing the operator $\mathscr{L}$ by one with a simpler spectrum. In this case $\mathscr{L}$ is a linear operator and the method of approximation is constructive and systematic. For nonlinear kinetic equations, as considered in the next section, the construction of kinetic models is more flexible and governed by the physical properties of interest. These methods and their relationship to other approximation schemes have been summarized recently in ref. 16. Briefly, in the linear case the operation of $\mathscr{L}$ on an arbitrary function f is expanded in a complete set of functions $\{\Psi_{\mathbf{n}}\}$, assumed to be orthonormalized with respect to an inner product indicated by (f, g),

$$\begin{aligned} \mathscr{L}f &= \sum_{\mathbf{n}} \Psi_{\mathbf{n}}(\mathbf{v})(\Psi_{\mathbf{n}}, \mathscr{L}f) \\ &= \sum_{\mathbf{n},\,\mathbf{m}} \Psi_{\mathbf{n}}(\mathbf{v})(\Psi_{\mathbf{n}}, \mathscr{L}\Psi_{\mathbf{m}})(\Psi_{\mathbf{m}}, f) = \left(\sum_{\mathbf{n},\,\mathbf{m}} \mathscr{P}_{\mathbf{n}} \mathscr{L} P_{\mathbf{m}}\right) f \end{aligned}$$

Here $\mathscr{P}_{\mathbf{n}}$ denotes the projection operator onto $\Psi_{\mathbf{n}}$. This provides a representation for the operator $\mathscr{L}$

$$\mathscr{L} = \sum_{\mathbf{n},\, \mathbf{m}} \mathscr{P}_{\mathbf{n}} \mathscr{L} \mathscr{P}_{\mathbf{m}} \tag{3.1}$$

Kinetic models are obtained by retaining the exact contributions from some finite set of projection operators, and approximating the contributions to $\mathscr{L}$ in the orthogonal subspace by a simple degenerate operator with a single eigenvalue (single relaxation time model). In the present case, the exact eigenfunctions are known and can be used as the basis set for the representation in Eq. (3.1). Then since $\mathscr{L}\mathscr{P}_{\mathbf{m}} = \lambda_{\mathbf{m}} \mathscr{P}_{\mathbf{m}}$, where $\lambda_{\mathbf{m}}$ is the eigenvalue associated with $\Psi_{\mathbf{m}}$ and $\mathscr{P}_{\mathbf{n}}\mathscr{P}_{\mathbf{m}} = \delta_{\mathbf{nm}} \mathscr{P}_{\mathbf{n}}$, Eq. (3.1) reduces to the spectral decomposition of $\mathscr{L}$,

$$\mathscr{L} = \sum_{\mathbf{n}} \lambda_{\mathbf{n}} \mathscr{P}_{\mathbf{n}} = 0\mathscr{P}_{\mathbf{0}} + \sum_{\mathbf{n} \neq \mathbf{0}} \lambda_{\mathbf{n}} \mathscr{P}_{\mathbf{n}} \tag{3.2}$$

The second equality makes explicit the fact that there is a null space, corresponding to the conservation of probability. The simplest kinetic model is obtained by collapsing the rest of the spectrum to a single degenerate point $-\nu$,

$$\mathscr{L} \to -\nu \sum_{\mathbf{n} \neq 0} \mathscr{P}_{\mathbf{n}} = \nu \mathscr{P}_{\mathbf{0}} - \nu \sum_{\mathbf{n}} \mathscr{P}_{\mathbf{n}} = -\nu(1 - \mathscr{P}_{\mathbf{0}}) \tag{3.3}$$

This approximation has a very simple interpretation. The null space associated with the conservation law is preserved but all relaxation in the orthogonal subspace is collapsed to a single infinitely degenerate relaxation rate ν. The corresponding model kinetic equation is

$$(\partial_t + \mathbf{v} \cdot \nabla)\, F = -\nu(F - n\psi_M) \tag{3.4}$$

where use has been made of the property $\mathscr{P}_{\mathbf{0}} F(\mathbf{r}, \mathbf{v}, t) = n(\mathbf{r}, t)\, \psi_M(\mathbf{v}/v_0)$, with ψ_M denoting the Maxwellian normalized to unity.

The approximation given by Eq. (3.4) is the usual BGK model, originally proposed by Bhatnagar, Gross, and Krook, specialized here to tagged particle dynamics.(5) It preserves the stationary state and normalization, while providing a simple single relaxation time description of the dynamics. Solutions to this equation are similar to those of the Fokker–Planck equation, showing a fast velocity relaxation on a time scale defined by ν^{-1} followed by a slower diffusion in space. The parameter ν can be chosen such that the equation for the average velocity is the same as that for the Fokker–Planck equation. In this case $\nu = \gamma_e(T_g)$. Application to calculate the self-structure function, measuring its accuracy for space and time dependence, shows both qualitative and semi-quantitative accuracy (10–15%) over the full space-time domain.[17]

The BGK kinetic model for elastic collisions corresponds to replacing all negative eigenvalues by the one with the smallest magnitude. For the case of *granular media* the analysis of the last section shows that elimination of the time dependent temperature for the bath by scaling the velocities generates an additional operator whose spectrum has arbitrarily large positive eigenvalues. Ultimately these are controlled by corresponding large eigenvalues of $\mathscr{L}$. If now the spectrum of $\mathscr{L}$ were collapsed to form a kinetic model without introducing a similar approximation for the effect of the time dependent temperature, the control of the latter by the former would no longer be possible, and unstable modes would appear in the solution to the kinetic model equation. Instead, the appropriate method is to collapse the spectrum of the combined operator $\mathscr{L}^{*\prime}$ defined in Eq. (2.35). Application of the usual kinetic modeling technique described above then gives

$$\mathscr{L}^{*\prime} = 0\mathscr{P}_{\mathbf{0}}^{*} + \sum_{\mathbf{n}\neq 0} \lambda_{\mathbf{n}}^{*\prime}\mathscr{P}_{\mathbf{n}}^{*} \rightarrow -(1-\varepsilon_0)\, \nu^*(1-\mathscr{P}_{\mathbf{0}}^{*}) \tag{3.5}$$

where ν^* and $\mathscr{P}_{\mathbf{0}}^{*}$ are the dimensionless forms of ν and $\mathscr{P}_{\mathbf{0}}$, respectively. The choice to use $\mathscr{L}^{*\prime}$ for constructing the kinetic model is suggested on physical as well as mathematical grounds, since its spectrum determines the modes or relaxation for the distribution function. For homogeneous states the model kinetic equation representing (2.32) becomes

$$\frac{\partial}{\partial t^*} F^* = -(1-\varepsilon_0)\, \nu^*(1-\mathscr{P}_{\mathbf{0}}^{*})\, F^* \tag{3.6}$$

In the original variables the model kinetic equation is

$$\partial_t F = -\nu(T_g)(1-\varepsilon_0)\left[F - \Omega^{-1}\psi_M\left(\frac{\mathbf{v}}{\hat{v}_0(t)}\right)\right] + \varepsilon_0 a_0(\alpha)\, \gamma_e(T_g)\, \frac{\partial}{\partial \mathbf{v}}\cdot(\mathbf{v}F) \tag{3.7}$$

The Maxwellian ψ_M with the time dependent scaling $\hat{v}_0(t)$ is again normalized to unity. The generalization to inhomogeneous states is straightforward,

$$(\partial_t + \mathbf{v}\cdot\nabla)\, F = -\nu(T_g)(1-\varepsilon_0)[F - n(\mathbf{r}, t)\, \psi_M(\mathbf{v}/\hat{v}_0(t))] + \varepsilon_0 a_0(\alpha)\, \gamma_e(T_g)\, \frac{\partial}{\partial \mathbf{v}}\cdot(\mathbf{v}F) \tag{3.8}$$

The HCS solution to this equation is $\Omega^{-1}\psi_M(\mathbf{v}/\hat{v}_0(t))$ which differs from Eq. (2.27) for F_H only by the replacement of the temperature with $T(t)\to T_g(t)\,a_0(\alpha)(1-\varepsilon_0)^{-1}$. As shown by Eq. (2.22), this difference vanishes after a short initial transient period. The parameter ν can be fixed by requiring that (3.8) yield the exact dynamics for the average velocity $\mathbf{u}(t)$, Eq. (2.17). This leads to the choice

$$\nu=a_0(\alpha)\,\gamma_e(T_g) \tag{3.9}$$

It can be shown that this choice also implies that the correct diffusion coefficient is predicted by the kinetic model. On the other hand, the exact evolution equation for the tagged particle temperature, Eq. (2.18), is obtained from the model only if $\nu=2a_0(\alpha)\,\gamma_e(T_g)$. The limitation of the model to reproduce exactly the evolution equation for more than one moment of the distribution function is a consequence of the presence of only a single free parameter, and occurs in the elastic case as well. In what follows, the choice (3.9) will be taken.

This completes the definition of the kinetic model. For $\alpha=\alpha_g=1$ it reduces to the BGK kinetic equation for a tagged particle. For the inelastic case it preserves the essential qualitative and semi-quantitative features of the Fokker–Planck equation. To illustrate this further, consider the general solution for homogeneous states

$$F(\mathbf{v},t)=\Omega^{-1}\psi_M(\mathbf{v}/\hat{v}_0(t))+e^{-(1-\varepsilon_0)\,t^*}[F_0(\mathbf{v},t)-\Omega^{-1}\psi_M(\mathbf{v}/\hat{v}_0(t))] \tag{3.10}$$

where t^* is the dimensionless time scale defined in Eq. (2.31) and $F_0(\mathbf{v},t)\equiv e^{d\varepsilon_0 t^*}F(e^{\varepsilon_0 t^*}\mathbf{v},0)$ is the initial distribution function with a scaled velocity corresponding to the change from $T(0)$ to $T(0)\,T_g(t)/T_g(0)$. The solution is positive definite for all times and monotonically decreases towards the HCS, just as the solution to the Fokker–Planck equation. The time scales for this relaxation are the same as well, although the details of the kinetic modes are of course different.

To analyze the solution to the kinetic model for inhomogeneous states it is again convenient to Fourier transform and use dimensionless variables

$$[\partial_{t^*}-i\mathbf{k}^*\cdot\mathbf{v}^*+\nu^*(1-\varepsilon_0)]\,\tilde{F}^*=\nu^*(1-\varepsilon_0)\,\tilde{n}^*(\mathbf{k}^*,t^*)\,\psi_M^* \tag{3.11}$$

with $\tilde{n}^*(\mathbf{k}^*,t^*)=\Omega\tilde{n}(\mathbf{k}^*,t^*)$ and $\psi_M^*(v^*)=\pi^{-d/2}e^{-v^{*2}}$. This is structurally the same as the Fourier transform of Eq. (3.4) for the elastic case in dimensionless variables, with an effective frequency $\nu(1-\varepsilon_0)/a_0(\alpha)\,\gamma_e(T_g)$. In these variables Eq. (3.11) can be solved by Laplace transform in the same way as for elastic fluids and all the known results for the latter case transfer directly to the case of inelastic collisions as well. This includes the existence of a hydrodynamic diffusive mode and a branch cut in the spectrum representing the infinitely many kinetic modes of the Fokker–Planck equation. The hydrodynamic mode is separated from the branch cut for all k^* less than a critical value $k_c^* = \sqrt{\pi}$. Thus, for long wavelength excitations the initial kinetic transients decay rapidly leaving the hydrodynamic mode as the dominant behavior. This is precisely the "aging to hydrodynamics" that characterizes the elastic case. However, it should be noted that this parallel between elastic and inelastic cases is in terms of the dimensionless variables. Qualitative differences occur when transforming back to the original variables where exponential decay in the dimensionless time maps to algebraic decay in real time.

The primary conclusions of the last two sections are that a Fokker–Planck equation with time dependent coefficients results from asymptotic analysis of the Boltzmann equation for a heavy tagged particle in a gas of lighter particles, where all collisions are inelastic. The exact solution to the Fokker–Planck equation shows both diffusion and kinetic modes in the appropriate dimensionless variables. The asymptotic solution for long times is a Maxwellian with time dependent temperature, which differs from that of the surrounding gas but with the same cooling rate. Analysis of the spectrum of the Fokker–Planck equation shows that it differs from that of the Fokker–Planck operator by a constant factor and the time dependence of the coefficients. Transformation of the velocity to dimensionless form eliminates this time dependence and introduces a modified Fokker–Planck operator whose spectrum describes the correct excitations. This analysis suggests the proper means to construct a kinetic model by contraction of the kinetic spectrum for the modified Fokker–Planck operator. The resulting kinetic model in dimensionless variables has the same form as for the elastic case. Previous studies of the latter show that the kinetic model equation is a good representation of the Fokker–Planck equation for both hydrodynamic and kinetic modes. In this way we have established the quantitative features of tagged particle motion in granular media and verified the applicability of kinetic models.

IV. KINETIC MODELS FOR THE BOLTZMANN EQUATION

In this section we exploit the above analysis of tagged particle motion to suggest how to construct kinetic models for the Boltzmann equation representing a one component gas with inelastic collisions. The Boltzmann equation for inelastic collisions is given by(2, 3)

$$(\partial_t + \mathbf{v} \cdot \nabla) f = J[f, f] \tag{4.1}$$

where the collision operator $J[f, f]$ is obtained from the functional (2.2) except that now both distribution functions in the integral are the unknown dependent variable f,

$$J[f, f] = \sigma^{d-1} \int d\mathbf{v}_1 \int d\hat{\boldsymbol{\sigma}}\, \Theta(\mathbf{g}\cdot\hat{\boldsymbol{\sigma}})(\mathbf{g}\cdot\hat{\boldsymbol{\sigma}}) \times [\alpha^{-2} f(\mathbf{r}, \mathbf{v}', t)\, f(\mathbf{r}, \mathbf{v}_1', t) - f(\mathbf{r}, \mathbf{v}, t)\, f(\mathbf{r}, \mathbf{v}_1, t)] \tag{4.2}$$

All quantities in this expression are defined as in Eq. (2.2) with the only differences that now $\Delta = 1$ and α is the coefficient of restitution for the gas particles. Then, the restituting velocities are given by

$$\mathbf{v}' = \mathbf{v} - \frac{1+\alpha}{2\alpha}(\mathbf{g}\cdot\hat{\boldsymbol{\sigma}})\,\hat{\boldsymbol{\sigma}}, \qquad \mathbf{v}_1' = \mathbf{v}_1 + \frac{1+\alpha}{2\alpha}(\mathbf{g}\cdot\hat{\boldsymbol{\sigma}})\,\hat{\boldsymbol{\sigma}} \tag{4.3}$$

The most important properties of $J[f, f]$ for the purposes here are those that determine the form of the macroscopic balance equations for mass, momentum, and energy,

$$\int d\mathbf{v} \begin{pmatrix} 1 \\ m\mathbf{v} \\ \frac{1}{2}mV^2 \end{pmatrix} J[f, f] = \begin{pmatrix} 0 \\ \mathbf{0} \\ -(d/2)\, nk_{\mathrm{B}} T\zeta \end{pmatrix} \tag{4.4}$$

where $\mathbf{V}(\mathbf{r}, t) = \mathbf{v} - \mathbf{u}(\mathbf{r}, t)$ and $\mathbf{u}$ is the nonequilibrium flow velocity. Also $\zeta(t)$ is the cooling rate already introduced in Eq. (2.6) and determined from the explicit form of $J[f, f]$ to be

$$\zeta = (1-\alpha^2)\frac{m\pi^{(d-1)/2}\sigma^{d-1}}{4d\Gamma((d+3)/2)\, nk_{\mathrm{B}} T} \int d\mathbf{v} \int d\mathbf{v}_1\, |\mathbf{v}-\mathbf{v}_1|^3 f(\mathbf{r}, \mathbf{v}, t)\, f(\mathbf{r}, \mathbf{v}_1, t) \tag{4.5}$$

As mentioned in Section II, a HCS of the form $f_H(v, t) = nv_0^{-d}(t)\, \phi(v/v_0(t))$ with $v_0(t) = [2k_{\mathrm{B}} T(t)/m]^{1/2}$ exists and it is determined by the solution to

$$(\partial_t T)\, \partial_T f_H = J[f_H, f_H] \tag{4.6}$$

or equivalently, using Eq. (2.6),

$$\frac{\zeta_H}{2}\frac{\partial}{\partial \mathbf{v}}\cdot(\mathbf{v} f_H) = J[f_H, f_H] \tag{4.7}$$

Equations (4.4) and (4.7) are the primary exact properties of the Boltzmann collision operator to be preserved in any acceptable kinetic model.

To construct the kinetic model recall from the previous section that the proper spectrum for the Fokker–Planck equation was obtained by scaling the velocities with a time dependent thermal velocity to account for cooling. The generator for the dynamics in this representation was not the scaled Fokker–Planck operator $\mathscr{L}^*$ but rather $\mathscr{L}^{*\prime} = \mathscr{L}^* - \varepsilon_0(\partial/\partial \mathbf{v}^*) \cdot \mathbf{v}^*$, where the second term represents the effects of cooling (see Eq. (3.5) and the discussion above it). Guided by these results for tagged particle motion, the Boltzmann equation is rewritten in the form

$$(\partial_t + \mathbf{v} \cdot \nabla) f - \frac{\zeta}{2} \frac{\partial}{\partial \mathbf{v}} \cdot (\mathbf{V} f) = J'[f, f] \tag{4.8}$$

$$J'[f, f] \equiv J[f, f] - \frac{\zeta}{2} \frac{\partial}{\partial \mathbf{v}} \cdot (\mathbf{V} f) \tag{4.9}$$

The velocity derivative term on the left side compensates for the time dependence in the HCS. The corresponding modification of $J[f, f]$ to $J'[f, f]$ on the right side leads to the properties

$$J'[f_H, f_H] = 0, \qquad \int d\mathbf{v} \begin{pmatrix} 1 \\ m\mathbf{v} \\ \frac{1}{2} m V^2 \end{pmatrix} J'[f, f] = \begin{pmatrix} 0 \\ \mathbf{0} \\ 0 \end{pmatrix} \tag{4.10}$$

as follows from direct evaluation using Eq. (4.4). Thus $J'[f, f]$ has the same structure as $J[f, f]$ for the elastic case: an invariant state and $d+2$ vanishing low velocity moments. This suggests that the usual BGK kinetic model for elastic collisions is appropriate to represent $J'[f, f]$ rather than $J[f, f]$ in the case of inelastic collisions. This is equivalent to collapsing the spectrum of J' instead of J. Therefore, we approximate

$$J'[f, f] \rightarrow -\nu(f - f_{\ell H}) \tag{4.11}$$

As in the case of the Fokker–Planck equation discussed in the previous section, ν has a space and time dependence that occurs only through the density and temperature, and is a free parameter of the model. Also, $f_{\ell H}$ is a local form of the homogeneous solution obtained from f_H by replacing $\mathbf{v} \rightarrow \mathbf{V} = \mathbf{v} - \mathbf{u}$, and by replacing the temperature and density with their local values for the nonequilibrium state considered. This implies that $f_{\ell h}$ has the same $d+2$ lowest velocity moments as f and assures that the kinetic model defined by Eq. (4.11) has the required properties (4.10). With this choice the model kinetic equation becomes

$$(\partial_t + \mathbf{v} \cdot \nabla) f = -\nu(f - f_{\ell H}) + \frac{1}{2} \zeta \frac{\partial}{\partial \mathbf{v}} \cdot (\mathbf{V} f) \tag{4.12}$$

This is the model we propose as a representation of the Boltzmann equation for inelastic collisions. By construction it yields the correct macroscopic balance equations for mass, energy, and momentum, and has the same HCS. The free parameter $\nu[n(\mathbf{r}t), T(\mathbf{r}, t)]$ can be chosen to optimize agreement with the viscosity or thermal conductivity for the Boltzmann equation. In these respects it has the same expected qualitative and quantitative features as the BGK equation for elastic collisions. A comparison of (4.12) with two closely related earlier kinetic models^(3, 6–8) has been presented elsewhere.^(8)

A. Solution to the Linearized Kinetic Equation

To illustrate the utility of the kinetic model, its linearization around the HCS is considered, and the solution for the initial value problem is given. But two more simplifications of the model will be introduced in this application. The function $f_{\ell H}$ must be determined from the solution f_H of the Boltzmann equation, Eq. (4.7). This problem is quite complex and an analytic form is not yet known. However, Monte Carlo simulations of the solution to this equation show the result remains quite close to a Maxwellian even for relatively large dissipation.^(18) Also, $\zeta[f]$ is the bilinear functional of f given in Eq. (4.5) and provides an additional complexity of the model kinetic equation not present for elastic collisions. Recent calculations of $\zeta[f]$ for weakly inhomogeneous states^(8) have shown that it is very well approximated by $\zeta \to \zeta_\ell \equiv \zeta[f_\ell]$, where f_ℓ is the local equilibrium state obtained from a Maxwellian. For practical purposes, therefore, two additional approximations in the kinetic model are considered in this section

$$f_{\ell H} \to f_\ell = n\left(\frac{m}{2\pi k_B T}\right)^{d/2} \exp\left(-\frac{mV^2}{2k_B T}\right) \tag{4.13}$$

$$\zeta[f] \to \zeta_\ell = (1-\alpha^2)\frac{2\pi^{(d-1)/2}\sigma^{d-1}}{d\Gamma(d/2)}\, n\left(\frac{k_B T}{m}\right)^{1/2} \tag{4.14}$$

leading to the final form

$$(\partial_t + \mathbf{v}\cdot\nabla) f = -\nu(f - f_\ell) + \frac{1}{2}\zeta_\ell \frac{\partial}{\partial \mathbf{v}}\cdot(\mathbf{V}f) \tag{4.15}$$

As a consequence of the approximations introduced by Eqs. (4.13) and (4.14), the energy balance equation and the homogeneous cooling solution are only approximately the same as for the Boltzmann equation. These are tolerable quantitative compromises that do not affect any qualitative relationship of the two kinetic equations. We emphasize that these additional approximations are not necessary for applications of the kinetic model.

The linearized form is obtained by defining the deviation of the distribution function from its homogeneous cooling form f_H by

$$\delta f = f - f_H \tag{4.16}$$

and the corresponding deviations of the hydrodynamic fields

$$\delta n = n - n_H, \qquad \delta T = T - T_H, \qquad \delta \mathbf{u} = \mathbf{u} \tag{4.17}$$

Consider an initial state for which the deviation δf is small and assume an interval of later times for which it remains small (a finite interval for unstable states). Then substitution of Eq. (4.16) into (4.15) and retaining only linear terms in δf gives

$$\begin{aligned}&\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla + \nu_H\right)\delta f - \frac{\zeta_H}{2}\frac{\partial}{\partial \mathbf{v}}\cdot(\mathbf{v}\delta f)\\ &= \frac{\delta n}{n_H}\left(\nu_H + \frac{\zeta_H}{2}\frac{\partial}{\partial \mathbf{v}}\cdot\mathbf{v}\right)f_H - \left(\nu_H + \frac{\zeta_H}{2}\right)\delta\mathbf{u}\cdot\frac{\partial}{\partial \mathbf{v}}f_H - \frac{\delta T}{2T_H}\left(\nu_H - \frac{\zeta_H}{2}\right)\frac{\partial}{\partial \mathbf{v}}\cdot(\mathbf{v}f_H)\end{aligned} \tag{4.18}$$

The detailed analysis of this linearized kinetic model is relegated to Appendix B. The possible excitations of the gas include the modes associated with the free particle motion modulated by pure exponential damping, the $d+2$ hydrodynamic modes, and the non-hydrodynamic, or kinetic, modes. The simplest illustration of these excitations occurs for the special case of initial transverse velocity perturbations. As described in Appendix B, the kinetic model provides a precise mapping of the macroscopic transverse velocity field for granular fluids to that for normal fluids. In particular, it shows that the known relaxation of kinetic modes to a long time, long wavelength hydrodynamic description is the same up to considerations of scaling. The results show that the transverse velocity field perturbations have two types of dynamical response. One is hydrodynamic with relaxation on a hydrodynamic time scale which diverges as the wavenumber goes to zero. The second is a fast kinetic relaxation on the time scale of the collision time. Thus for long times and long wavelength perturbations the hydrodynamic relaxation dominates and a simpler hydrodynamic description is justified.

The analysis is similar for the response to more general initial perturbations. Although there is not an exact scaling relation between the elastic and inelastic cases, as in Eq. (B.29), it still follows that there is a separation into short time kinetic and long wavelength hydrodynamic excitations. The hydrodynamic excitations include the above $d-1$ shear modes, plus three additional modes. For elastic collisions the latter are the two sound modes and the heat diffusion mode. In the case of inelastic collisions these three modes are more complex and do not necessarily have this same physical interpretation. The hydrodynamic description will be presented in detail elsewhere.

B. Uniform Shear Flow

To illustrate an application of the *nonlinear* kinetic model consider the case of uniform shear flow, characterized by a uniform temperature and density, and a flow field $u_x = ay$. The stationary distribution function has the form $f(\mathbf{r}, \mathbf{v}; t) = f_s(\mathbf{V})$ and is the solution to

$$-a_{ij} V_j \frac{\partial}{\partial V_i} f_s - \frac{\partial}{\partial \mathbf{V}} \cdot \left(\frac{1}{2} \zeta_\ell \mathbf{V} f_s \right) = -\nu (f_s - f_\ell) \qquad (4.19)$$

with $a_{ij} = a\delta_{ix}\delta_{jy}$. Note we are using again the simplified version of the model with the two additional approximations given by Eqs. (4.13) and (4.14). Solutions of Eq. (4.19) exist only for special values of ζ_ℓ, determined by the condition that the temperature should be constant, i.e.,

$$\frac{dn_s k_{\mathrm{B}} T_s}{2} \zeta_\ell(n_s, T_s) = \eta(a, T_s)\, a^2, \qquad \eta(a)\, a = -\int d\mathbf{V}\, m V_x V_y f_s(\mathbf{V}) \qquad (4.20)$$

The second equation identifies $\eta(a)$ as the shear rate dependent shear viscosity. The first equation defines a special temperature, $T_s(a)$, at which the stationary state occurs. This represents the balance between viscous heating due to the imposed shear and cooling due to the inelastic collisions. For the case of elastic collisions, the latter effect can be imposed by the addition of a thermostat obtained using an external non-conservative force of the form $\mathbf{F}_{\mathrm{ext}} = -\lambda(a)\, m\mathbf{V}$. The corresponding stationary kinetic equation has the same form as (4.19) with the correspondence $\zeta_\ell \leftrightarrow 2\lambda(a)$. With this identification, all the known results for uniform shear flow with elastic collisions and a thermostat$^{(19)}$ can be transferred to the case of inelastic collisions. For example, the exact stationary solution to (4.19) is

$$f_s(\mathbf{V}) = \int_0^\infty dt\, e^{-t(1-(d/2)\zeta^*)} f_\ell(e^{(1/2)\zeta^* t} \Lambda_{ij}(-t)\, V_j), \qquad \Lambda_{ij}(t) = \delta_{ij} - (a_{ij}/\nu_s)\, t \qquad (4.21)$$

where $\zeta^* \equiv \zeta_\ell(n_s, T_s)/\nu(n_s, T_s)$ is independent of the temperature and shear rate, but otherwise depends on the coefficient of restitution; in the three-dimensional case, $\zeta^* = 5(1-\alpha^2)/(4+6\alpha+2\alpha^2)$. The solution depends on the shear rate through the explicit dependence in $\Lambda_{ij}(t)$, and through the dependence of f_ℓ and ν_s on $T_s(a)$. The latter is determined by taking the second moments of (4.19) to get

$$\nu_s^2 \zeta^* (1+\zeta^*)^2 = \frac{2}{d}\, a^2 \tag{4.22}$$

Let T_0 denote some constant reference temperature, e.g., the initial temperature when the shear is applied. Also, define $\nu_0 \equiv \nu(n_s, T_0)$, $T^*(a) \equiv T_s(a)/T_0$, and $a^* \equiv a/\nu_0$. Then (4.22) gives

$$T^*(a) = \frac{2}{d\zeta^*}\, \frac{a^{*2}}{(1+\zeta^*)^2} \tag{4.23}$$

This completely determines the shear rate dependence of the exact solution (4.21). Any property of interest can be calculated from this solution by quadratures. For example, the shear viscosity is readily found from (4.21) to be

$$\eta(a, T_s) = \frac{p_s}{\nu_s (1+\zeta^*)^2} \tag{4.24}$$

The shear rate dependence of the viscosity occurs entirely through the temperature. Since $p_s/\nu_s \propto \sqrt{T_s(a)}$ this result implies $\eta(a, T_s) \propto a$. Other properties such as the viscometric functions can be calculated simply in a similar way. For instance, $P_{xx} - P_{yy} = p_s d\zeta^*/(1+\zeta^*)$.

This example of shear flow shows the value of the kinetic model for insight about nonlinear transport in states far from equilibrium that would be difficult to explore directly from the Boltzmann equation. Comparisons in the elastic case between Monte Carlo simulation of the Boltzmann equation and the kinetic model show good agreement except at very large shear rates.(20) Recent simulations for the inelastic case show a similar good agreement.(18, 21, 22)

V. KINETIC MODELS AT FINITE DENSITY

The previous discussion has been restricted to the case of a low density gas for which the Boltzmann description is presumed appropriate. At higher densities the revised Enskog kinetic theory (RET) is known to provide an accurate description for elastic collisions.(9) Its generalization to inelastic collisions is straightforward and provides the basis for analysis of granular flow at finite densities. The RET kinetic equation is given by$^{(3)}$

$$\left(\frac{\partial}{\partial t}+\mathbf{v}\cdot\nabla\right) f(\mathbf{r}, \mathbf{v}; t)=J_E[\mathbf{r}, \mathbf{v} \mid f(t)] \tag{5.1}$$

$$\begin{aligned} J_E[\mathbf{r}, \mathbf{v} \mid f(t)] = \sigma^{d-1} \int d\mathbf{v}_1 \int d\hat{\boldsymbol{\sigma}}\, \Theta(\mathbf{g}\cdot\hat{\boldsymbol{\sigma}})(\mathbf{g}\cdot\hat{\boldsymbol{\sigma}}) \\ \times \{\alpha^{-2}\chi[\mathbf{r}, \mathbf{r}-\boldsymbol{\sigma} \mid n(t)]\, f(\mathbf{r}, \mathbf{v}', t)\, f(\mathbf{r}-\boldsymbol{\sigma}, \mathbf{v}_1', t) \\ -\chi[\mathbf{r}, \mathbf{r}+\boldsymbol{\sigma} \mid n(t)]\, f(\mathbf{r}, \mathbf{v}, t)\, f(\mathbf{r}+\boldsymbol{\sigma}, \mathbf{v}_1, t)\} \end{aligned} \tag{5.2}$$

The notation is the same as in (4.2) and (4.3). The differences in the spatial arguments on the right side represent the fact that a colliding pair of particles have their centers separated by $\pm\boldsymbol{\sigma}=\pm\sigma\hat{\boldsymbol{\sigma}}$. Also, $\chi[\mathbf{r}_1, \mathbf{r}_2 \mid n(t)]$ is the pair correlation function for an equilibrium system with non-uniform density field $n(\mathbf{r}, t)$. It gives the probability to find particles at $\mathbf{r}_1$ and $\mathbf{r}_2$ for a system at equilibrium in an external potential $U_{\text{ext}}(\mathbf{r})$ chosen to produce the density field of the actual nonequilibrium state

$$n(\mathbf{r}, t)=\int d\mathbf{v}\, f(\mathbf{r}, \mathbf{v}, t) \tag{5.3}$$

Thus, $\chi[n]$ is an *equilibrium* functional that can be determined exactly from the second functional derivative of the equilibrium free energy functional for an inhomogeneous state, but it is evaluated at the nonequilibrium density. This implies that $\chi[n]$ and, consequently, $J_E[f]$ are highly non-linear functionals of f through this density dependence and Eq. (5.3).

The construction of a kinetic model with the same qualitative features as the RET has been discussed recently$^{(10, 11)}$ and we review briefly the proposed form in the case of elastic collisions. A primary new feature of the RET collision operator is collisional transfer contributions to the heat and momentum fluxes. As a consequence, there are additional contributions to the right side of (4.4) for the moments of $J_E[f]$ representing these collisional transfer terms. To account for this important effect, the projection of $J_E[f]$ onto these moments is extracted explicitly,

$$J_E[f]=\mathscr{P}J_E[f]+(1-\mathscr{P})\, J_E[f] \tag{5.4}$$

where $\mathscr{P}$ is the projection operator onto an orthonormal set constructed from $\{1, v^2, \mathbf{v}\}$:

$$\mathscr{P}g(\mathbf{v}) = n^{-1} \sum_{\alpha} \psi_\alpha(\mathbf{v})\, f_{\ell H}(\mathbf{v}) \int d\mathbf{v}'\, \psi_\alpha(\mathbf{v}')\, g(\mathbf{v}') \tag{5.5}$$

$$\{\psi_\alpha\} = \left\{1,\, c^{-1/2}\left(\frac{m\beta}{2} V^2 - \frac{d}{2}\right),\, (m\beta)^{1/2}\,\mathbf{V}\right\} \tag{5.6}$$

with the normalization coefficient

$$c = n^{-1} \int d\mathbf{v}\, f_{\ell H}(\mathbf{v}) \left(\frac{m\beta}{2} V^2 - \frac{d}{2}\right)^2 \tag{5.7}$$

where $\beta \equiv 1/k_{\mathrm{B}} T$. The first term on the right hand side of Eq. (5.4) gives the collisional transfer contributions to the fluxes in the conservation equations, and must be retained in any acceptable kinetic model. The second term on the right side does not contribute to the form of the conservation laws and the simplest approximation is to represent $J_E[f]$ in this "less important" subspace as the negative of an effective collision frequency, λ, times the distribution function

$$J_E[f] \to \mathscr{P} J_E[f] - (1 - \mathscr{P})\, \lambda f \tag{5.8}$$

This choice retains the important qualitative features the RET regardless of the choice for λ: exact conservation laws with the correct fluxes, and the exact stationary states (both fluid and crystal). Further details of the reduction of this model and the choice for λ are given in ref. 11.

From the analysis of Sections III and IV, it is clear that the extension of this kinetic model to inelastic collisions should be implemented in terms of $J'_E[f] = J_E[f] - \frac{1}{2}\zeta\, \partial/\partial\mathbf{v} \cdot (\mathbf{V}f)$:

$$J'_E[f] \to \mathscr{P} J'_E[f] - (1 - \mathscr{P})\, \lambda f \tag{5.9}$$

The resulting model kinetic equation is then found to be (see Appendix C)

$$\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right) f = -\nu(f - f_{\ell H}) + \frac{1}{2}\zeta \frac{\partial}{\partial \mathbf{v}} \cdot (\mathbf{V}f) + \mathscr{I}[\mathbf{r}, \mathbf{v} \mid f] \tag{5.10}$$

This has the same form as (4.12) except for the additional functional on the right side,

$$\begin{aligned}\mathscr{I}[\mathbf{r}, \mathbf{v} \mid f] = {} & -\frac{\beta}{n} f_{\ell H}\Bigg[\mathbf{V}\nabla : \mathsf{P}^c[\mathbf{r}, \mathbf{v} \mid f] \\ & + \frac{d}{2c}\left(\frac{m\beta}{d} V^2 - 1\right)(\nabla\cdot\mathbf{q}^c[\mathbf{r}, \mathbf{v} \mid f] + \mathsf{P}^c[\mathbf{r}, \mathbf{v} \mid f] : \nabla\mathbf{u})\Bigg] \\ & + f_{\ell H}[\mathsf{A}[\mathbf{r}, \mathbf{v} \mid f] : \mathsf{D}(\mathbf{V}) + \mathbf{B}[\mathbf{r}, \mathbf{v} \mid f]\cdot\mathbf{S}(\mathbf{V})]\end{aligned} \tag{5.11}$$

with

$$\mathsf{D}(\mathbf{V}) = m\left(\mathbf{VV} - \frac{1}{d}V^2\mathbb{1}\right), \qquad \mathbf{S}(\mathbf{V}) = \left(\frac{m}{2}V^2 - \frac{2}{d\beta}\left(c + \frac{d^2}{4}\right)\right)\mathbf{V} \tag{5.12}$$

$\mathbb{1}$ being the unit tensor. The functionals $\mathsf{P}^c[\mathbf{r}, \mathbf{v} \mid f]$, $\mathbf{q}^c[\mathbf{r}, \mathbf{v} \mid f]$, $\mathsf{A}[\mathbf{r}, \mathbf{v} \mid f]$, and $B[\mathbf{r}, \mathbf{v} \mid f]$ are determined from moments of $J_E[f]$ and are given more explicitly in Appendix C. They vanish at low density limit so that the kinetic model (4.12) for the Boltzmann equation is regained in this limit.

VI. DISCUSSION

The objective here has been to present a kinetic model for the Boltzmann equation (and its high density counterpart, the RET equation) for a system of hard spheres with inelastic collisions. For elastic collisions, the corresponding kinetic models have provided analysis of dynamical phenomena far beyond that for which the Boltzmann equation is tractable and a similar role is expected for the inelastic case as well. There are three main components to our presentation: an exact analysis of tagged particle motion for a heavy particle in a dilute gas, construction of kinetic models based on a spectral analysis of effects due to cooling, and illustration of their application in two non-trivial examples beyond the practical reach of the Boltzmann equation. We summarize and comment on the primary results:

- The Boltzmann–Lorentz equation was used to describe tagged particle motion in a dilute gas of inelastic hard spheres. The restitution coefficients for tagged particle-gas particle collisions and that for the gas particles were taken to be independent. This kinetic equation was reduced to the corresponding Fokker–Planck equation using a Kramers–Moyal expansion in the limit of small $m_g T(t)/m T_g(t)$. In addition to small mass ratio, a new condition on the Fokker–Planck limit occurs due to the cooling of the gas. It is required that $T(t)/T_g(t)$ not grow without bound for long times or, equivalently, that the cooling rate of the gas $\zeta[T_g(t)]$ not exceed the relaxation rate of the tagged particle $2a_0(\alpha)\,\gamma_e[T_g(t)]$. This implies that the deviation of the restitution coefficient for the gas particle collisions from the elastic limit is only of order m_g/m. However, the relevant time scales for the tagged particle motion are of order $\sqrt{m/m_g}$ and the gas cooling effects are significant on this time scale. These conditions can be satisfied without any restrictions on the restitution coefficient for the tagged particle-gas collisions. The Fokker–Planck equation can be solved exactly for arbitrary initial conditions and describes velocity relaxation to a hydrodynamic (diffusive) stage and asymptotic approach to a Maxwellian distribution with time dependent temperature. The cooling rate of the particle in this asymptotic state is the same as that of the surrounding gas, but in general the temperatures of the gas and tagged particle are different. The approach to this asymptotic state is characterized by an infinite set of discrete modes, just as in the elastic case, but these modes are shifted a finite amount due to the effects of the gas cooling. Furthermore, these modes are not determined from eigenvalues of the Fokker–Planck operator but rather a related operator obtained by transformation to dimensionless variables to remove the time dependent temperature. On the basis of this last observation regarding the relevant spectrum, a model kinetic equation was constructed whose solution is in good agreement with that for the Fokker–Planck equation. The primary conclusions from this analysis are that the qualitative features of fast relaxation to a dominant hydrodynamic stage can be justified for systems with inelastic collisions and that these features can be captured quantitatively in a simple kinetic model.

- The Boltzmann equation for a low density gas of inelastic hard spheres was rewritten to identify the relevant collision operator $J'[f, f] \equiv J[f, f] - \frac{1}{2}\zeta\, \partial/\partial \mathbf{v} \cdot (\mathbf{V} f)$ accounting for the effects of cooling. The corresponding model kinetic equation was obtained by replacing $J'[f, f]$ with a single relaxation time form. The resulting kinetic model yields the same macroscopic balance equations for mass, energy, and momentum as those obtained from the Boltzmann equation. In addition, it supports the same HCS solution. In this respect, the model kinetic equation retains all the physical mechanisms relevant for inelastic collisions and a possible fluid dynamics description. Two examples were provided to demonstrate its utility under conditions for which the Boltzmann equation is not tractable. The first is an exact solution to the linearized equation for states close to the HCS. The solution allows complete characterization of the spectrum, including both fast kinetic modes and slower hydrodynamic modes. The case of an initial shear excitation was considered in detail, where an exact mapping of the spectrum for inelastic collisions to that for elastic collisions was obtained. In this way the approach to a hydrodynamic stage was again demonstrated, without restriction to weak inelasticity or to the Navier–Stokes long wavelength limit. The second example was a gas undergoing uniform shear flow. It was observed that the stationary solution to the nonlinear kinetic equation is equivalent to that for a gas with elastic collisions and a thermostat. The parameters of the thermostat can be related simply to the cooling rate so that the solution known for the elastic case can be adopted for the inelastic case as well. The exact solution and shear rate dependent viscosity were identified in this way for arbitrary values of the shear rate, including states very far from equilibrium.

- The method for constructing kinetic models from the Boltzmann equation was extended to the dense fluid revised Enskog kinetic equation. Recent applications of such a dense fluid model for shear flow in the elastic case show excellent agreement with molecular dynamics and Monte Carlo simulation results.(8) Preliminary calculations for the inelastic case show a similar accuracy.(23)

- The analysis of the Boltzmann–Lorentz equation considered the special case for which the background gas is in its homogeneous cooling state. It is known that this state is unstable to long wavelength perturbations(24) so the results obtained apply only over time scales short compared to that for growth of perturbations. However, as described in Section II, the conditions of the derivation require $\alpha_g \to 1$ as well as $\Delta \to 0$. Consequently, the background gas is asymptotically stable and the analysis is not compromised in any way by this complication. Regarding the remainder of the paper, the kinetic models are parameterized by the *local* cooling distribution which is not homogeneous. In fact, it describes the exact spatial variations of the hydrodynamic fields. Consequently, at the level of the kinetic equation the instability is expected to be well-described and is confirmed by Monte Carlo simulations. The Chapman–Enskog expansion of the kinetic theory above the homogeneous cooling state to obtain hydrodynamics is restricted by the instability. There are two different cases. In the first one, the time scale for the instability is long compared to that for the establishment of hydrodynamics, and the hydrodynamic equations in fact provide a means for the analysis of the onset and growth of the instability.(25) A second case is extreme dissipation such that the homogeneous cooling state becomes unstable prior to the dominance of a hydrodynamic description based on that state. This does not rule out a closed description based on the hydrodynamic fields obtained from expansion about a different reference state. This possibility has not been explored to date, but the kinetic models presented here provide an appropriate basis for considering such complex states. The formation of high density clusters would invalidate the kinetic model based on the Boltzmann equation, but that based on the RET in Section V. is particularly promising for such states since they are well-described through the pair functional $\chi[\mathbf{r}_1, \mathbf{r}_2 \mid n(t)]$ (note that the RET for elastic collisions supports fluid, solid, and coexisting states).

In summary, practical kinetic models for exploring the qualitative and quantitative features of inelastic hard spheres have been proposed, analyzed, and illustrated. They are expected to provide a new basis for exploring states far from equilibrium and boundary driven systems in a complementary and more detailed fashion than via fluid dynamics, and under conditions for which the latter may not be applicable.

APPENDIX A. DERIVATION OF THE FOKKER–PLANCK EQUATION

In this appendix a formal expansion of Eq. (2.1) in $\Delta = m_g/m$ is described. To set up the expansion, it is convenient to represent the collision operator J in terms of its adjoint. Let $H(\mathbf{v})$ be an arbitrary function of $\mathbf{v}$ and consider the integral

$$\begin{aligned} I[H] &= \int d\mathbf{v}\, H(\mathbf{v})\, J[\mathbf{v} \mid F, f] \\ &= \sigma_0^{d-1} \int d\mathbf{v} \int d\mathbf{v}_1\, H(\mathbf{v}) \\ &\quad \times \int d\hat{\boldsymbol{\sigma}}\, \Theta(\mathbf{g}\cdot\hat{\boldsymbol{\sigma}})(\mathbf{g}\cdot\hat{\boldsymbol{\sigma}})[\alpha^{-2} F(\mathbf{v}')\, f(\mathbf{v}_1') - F(\mathbf{v})\, f(\mathbf{v}_1)] \end{aligned} \tag{A.1}$$

The space and time dependence of the functions have been omitted for simplicity. We change variables in the first term on the right hand side to get

$$I[H] = \sigma_0^{d-1} \int d\mathbf{v} \int d\mathbf{v}_1\, F(\mathbf{v})\, f(\mathbf{v}_1) \int d\hat{\boldsymbol{\sigma}}\, \Theta(\mathbf{g}\cdot\hat{\boldsymbol{\sigma}})(\mathbf{g}\cdot\hat{\boldsymbol{\sigma}})[H(\mathbf{v}-\delta\mathbf{v}) - H(\mathbf{v})] \tag{A.2}$$

where

$$\delta\mathbf{v} = \frac{(1+\alpha)\,\Delta}{1+\Delta}\,(\mathbf{g}\cdot\hat{\boldsymbol{\sigma}})\,\hat{\boldsymbol{\sigma}} \tag{A.3}$$

Assuming Δ is small, a Kramers–Moyal expansion in the velocity jumps $\delta\mathbf{v}$ is performed to second order,

$$I[H] \simeq \int d\mathbf{v}\, H(\mathbf{v}) \left\{ \frac{\partial}{\partial\mathbf{v}} \cdot [\mathbf{A}(\mathbf{v})\, F(\mathbf{v})] + \frac{1}{2} \frac{\partial}{\partial\mathbf{v}} \frac{\partial}{\partial\mathbf{v}} : [\mathsf{N}(\mathbf{v})\, F(\mathbf{v})] \right\} \tag{A.4}$$

with

$$\mathbf{A}(\mathbf{v}) = \frac{(1+\alpha)\,\Delta}{1+\Delta}\,\frac{\pi^{(d-1)/2}\sigma_0^{d-1}}{\Gamma((d+3)/2)} \int d\mathbf{v}_1 f(\mathbf{v}_1)\, g\mathbf{g} \tag{A.5}$$

$$\mathsf{N}(\mathbf{v}) = \left[\frac{(1+\alpha)\,\Delta}{1+\Delta}\right]^2 \frac{\pi^{(d-1)/2}\sigma_0^{d-1}}{\Gamma((d+5)/2)} \times \int d\mathbf{v}_1 f(\mathbf{v}_1) \left[\frac{d+3}{2d}\, g^3 \mathbb{1} + \frac{3}{2}\, g \left(\mathbf{g}\mathbf{g} - \frac{1}{d}\, g^2 \mathbb{1}\right)\right] \tag{A.6}$$

Here $\mathbb{1}$ is the second order unit tensor. Since $H(\mathbf{v})$ is an arbitrary function, comparison of Eqs. (A.1) and (A.4) gives

$$J[\mathbf{v} \mid F, f] \simeq \frac{\partial}{\partial \mathbf{v}} \cdot [\mathbf{A}(\mathbf{v})\, F(\mathbf{v})] + \frac{1}{2}\frac{\partial}{\partial \mathbf{v}}\frac{\partial}{\partial \mathbf{v}} : [\mathsf{N}(\mathbf{v})\, F(\mathbf{v})] \tag{A.7}$$

where terms neglected are at least of order Δ^3. To identify the remaining Δ dependence of $\mathbf{A}$ and N, we particularize for the case of a gas in its HCS and introduce the thermal velocities

$$\tilde{\mathbf{v}} = \frac{\mathbf{v}}{v_0(t)}, \qquad \tilde{\mathbf{v}}_1 = \frac{\mathbf{v}_1}{v_g(t)} \tag{A.8}$$

where $v_g(t)$ was defined following Eq. (2.4) and $v_0(t) = [2k_B T(t)/m]^{1/2}$, with $T(t)$ being the temperature parameter for the tagged particle, defined in Eq. (2.15). We also introduce

$$\tilde{\mathbf{g}} = \frac{\mathbf{g}}{v_0(t)} = \tilde{\mathbf{v}} - \left[\frac{T_g(t)}{T(t)}\right]^{1/2} \Delta^{-1/2}\, \tilde{\mathbf{v}}_1 \tag{A.9}$$

The idea is that the relevant velocities in determining the properties of the tagged particle and their time evolution are of the order of the thermal velocities for both the tagged particle and the gas ones. Notice that there is no reason a priori to assume that $T_g(t)$ and $T(t)$ are of the same order. In fact, we will see below that this is not always the case. A formal expansion in $(T/T_g)\,\Delta$ leads to

$$\mathbf{A}(\mathbf{v}, t) \simeq \gamma(t)\, \mathbf{v}, \qquad \mathsf{N}(\mathbf{v}, t) \simeq 2\bar{\gamma}(t)\, \mathbb{1} \tag{A.10}$$

where

$$\gamma(t) = (1+\alpha)\, \frac{2\pi^{(d-1)/2}\sigma_0^{d-1}}{\Gamma((d+1)/2)\, d}\, n_g \left(\frac{2k_B T_g}{m}\right)^{1/2} \Delta^{1/2} \int d\mathbf{v}\, \phi(v)\, v \tag{A.11}$$

$$\bar{\gamma}(t) = (1+\alpha)^2\, \frac{\pi^{(d-1)/2}\sigma_0^{d-1}}{2\Gamma((d+3)/2)\, d}\, n_g \left(\frac{2k_B T_g}{m}\right)^{3/2} \Delta^{1/2} \int d\mathbf{v}\, \phi(v)\, v^3 \tag{A.12}$$

In the elastic limit, $\alpha = \alpha_g = 1$, the HCS becomes the usual equilibrium state with constant temperature and $\phi(v) = \pi^{-d/2} e^{-v^2}$. Then, Eqs. (A.11) and (A.12) reduce to

$$\gamma_e(T_g) = \frac{4\pi^{(d-1)/2}\sigma_0^{d-1} n_g \Delta^{1/2}}{d\Gamma(d/2)} \left(\frac{2k_B T_g}{m}\right)^{1/2}, \quad \bar{\gamma}_e(T_g) = \frac{k_B T_g}{m}\, \gamma_e(T_g) \tag{A.13}$$

This last equality is the expected fluctuation-dissipation relation. Returning to the HCS for a granular gas and an inelastic tagged particle, we can rewrite Eqs. (A.11) and (A.12) in the form

$$\gamma(t) = \gamma_e[T_g(t)]\, a(\alpha) \tag{A.14}$$

$$\bar{\gamma}(t) = \gamma_e[T_g(t)]\, a(\alpha)\, b(\alpha)\, \frac{k_B T_g(t)}{m} \tag{A.15}$$

where we have introduced the functions

$$a(\alpha) = \frac{(1+\alpha)\, \Gamma(d/2)}{2\Gamma((d+1)/2)} \int d\mathbf{v}\, \phi(v)\, v \tag{A.16}$$

$$b(\alpha) = \frac{1+\alpha}{d+1}\, \frac{\int d\mathbf{v}\, \phi(v)\, v^3}{\int d\mathbf{v}\, \phi(v)\, v} \tag{A.17}$$

The reduced distribution function for the gas particles in the HCS has a weak dependence on the coefficient of restitution α_g. This dependence is associated with the deviation of $\phi(v)$ from the Maxwellian form $\pi^{-d/2} e^{-v^2}$. In the elastic limit for the whole system, $\alpha = \alpha_g = 1$, both functions $a(\alpha)$ and $b(\alpha)$ reduce to unity.

Use of Eqs. (A.7) and (A.10) into Eq. (2.1) leads to an equation of the Fokker–Planck form,

$$(\partial_t + \mathbf{v}\cdot\nabla)\, F(\mathbf{r}, \mathbf{v}, t) = \gamma_e[T_g(t)]\, a(\alpha)\, \frac{\partial}{\partial \mathbf{v}} \cdot \left[\mathbf{v} + \frac{k_B T_g(t)}{m}\, b(\alpha)\, \frac{\partial}{\partial \mathbf{v}}\right] F(\mathbf{r}, \mathbf{v}, t) \tag{A.18}$$

To clarify the context of the Fokker–Planck limit in which the above equation has been derived, it is convenient to consider the time evolution of the temperature parameter $T(t)$ of the tagged particle. Equation (A.18) leads to

$$[\partial_t + 2\gamma_e(T_g)\, a(\alpha)]\, T(t) = 2\gamma_e(T_g)\, a(\alpha)\, b(\alpha)\, T_g(t) \tag{A.19}$$

that using Eq. (2.6) can be rewritten as

$$[\partial_t + 2\gamma_e(T_g)\, a(\alpha) - \zeta(T_g)]\, \frac{T(t)}{T_g(t)} = 2\gamma_e(T_g)\, a(\alpha)\, b(\alpha) \tag{A.20}$$

The general solution of this equation is

$$\frac{T(t)}{T_g(t)} = e^{-2(1-\varepsilon)\, t^*}\, \frac{T(0)}{T_g(0)} + b(\alpha)(1-\varepsilon)^{-1}\, [1 - e^{-2(1-\varepsilon)\, t^*}] \tag{A.21}$$

where t^* is the dimensionless time scale defined by

$$t^*(t) = a(\alpha) \int_0^t dt'\, \gamma_e[T_g(t')] \tag{A.22}$$

and

$$\varepsilon = \frac{\zeta[T_g(t)]}{2a(\alpha)\, \gamma_e[T_g(t)]} \tag{A.23}$$

Since both ζ and γ_e are proportional to $T_g^{1/2}$, ε is a time independent constant. From Eq. (A.21) it follows that the asymptotic behavior of the ratio $T(t)/T_g(t)$ for $t \to \infty$ ($t^* \to \infty$) depends quite strongly on the value of ε. For $\varepsilon < 1$, the ratio becomes a constant given by

$$\lim_{t \to \infty} \frac{T(t)}{T_g(t)} = b(\alpha)(1-\varepsilon)^{-1} \tag{A.24}$$

while for $\varepsilon \geqslant 1$ it grows without bound. This behavior is easily understood since ε is a measure of the cooling rate of the gas as compared to the temperature relaxation rate for the tagged particle.

Taking into account that the derivation of the Fokker–Planck equation requires the combination $(T/T_g)\, \Delta$ to be small, it is clear that it is restricted to systems with $\varepsilon < 1$. Closer inspection of the detailed expressions for $\zeta(T_g)$, $\gamma_e(T_g)$, and $a(\alpha)$ shows that this condition implies the limits

$\Delta \to 0$ and $\alpha_g \to 1$, scaled in the form given by Eq. (2.14). In this way, Eqs. (2.10) and (2.11) follow from Eq. (A.20) by using

$$\lim_{\alpha_g \to 1} b(\alpha) = \lim_{\alpha_g \to 1} a(\alpha) = \frac{1+\alpha}{2} \tag{A.25}$$

APPENDIX B. DETAILS OF THE SOLUTION TO THE LINEARIZED KINETIC EQUATION

In this appendix we analize in detail the linearized kinetic equation (4.18). First, it is useful to introduce the dimensionless variables

$$\mathbf{v}^* = \frac{\mathbf{v}}{v_H(t)}, \qquad \mathbf{r}^* = \frac{\nu_H(t)}{v_H(t)}\,\mathbf{r}, \qquad dt^* = \nu_H(t)\,dt \tag{B.1}$$

and the dimensionless quantities

$$\zeta^* = \frac{\zeta_H}{\nu_H}, \qquad \delta f^* = \frac{v_H^d(t)}{n_H}\,\delta f \tag{B.2}$$

with $v_H(t) = v_0[T_H(t)]$. The linearized kinetic model equation then becomes

$$\left(\frac{\partial}{\partial t^*} + \mathbf{v}^* \cdot \frac{\partial}{\partial \mathbf{r}^*} + 1\right) \delta f^*(\mathbf{r}^*, \mathbf{v}^*, t^*) = f_H^*(\mathbf{v}^*) \sum_i C_i(\mathbf{r}^*, t^* \mid \delta f^*)\, \psi_i(\mathbf{v}^*) \tag{B.3}$$

where the functions $C_i(\mathbf{r}^*, t^* \mid \delta f^*)$ are the normalized deviations of the hydrodynamic fields from their values in the homogeneous state,

$$\{C_i\} \equiv \{C_1, \mathbf{C}, C_{d+2}\} = \left\{\frac{\delta n}{n_H}, 2^{1/2}\,\frac{\delta \mathbf{u}}{v_H(t)}, \left(\frac{d}{2}\right)^{1/2} \frac{\delta T}{T_H}\right\} \tag{B.4}$$

They are dimensionless linear functionals of δf^* given by

$$C_i(\mathbf{r}^*, t^* \mid \delta f^*) = \int d\mathbf{v}^*\, \chi_i(\mathbf{v}^*)\, \delta f^*(\mathbf{r}^*, \mathbf{v}^*, t^*) \tag{B.5}$$

with

$$\{\chi_i(\mathbf{v}^*)\} = \left\{1, 2^{1/2}\mathbf{v}^*, \left(\frac{d}{2}\right)^{1/2} \left(\frac{2}{d}\, v^{*2} - 1\right)\right\} \tag{B.6}$$

The functions χ_i are an orthonormal set with respect to integration over the normalized Gaussian $f_H^*(\mathbf{v}^*)$,

$$\int d\mathbf{v}^* \, f_H^*(\mathbf{v}^*) \, \chi_i(\mathbf{v}^*) \, \chi_j(\mathbf{v}^*) = \delta_{ij} \tag{B.7}$$

Finally, the associated functions $\psi_i(\mathbf{v}^*)$ in (B.3) are

$$\{\psi_i(\mathbf{v}^*)\} = \left\{1 - \zeta^* \left(\frac{d}{2}\right)^{1/2} \chi_{d+2}(\mathbf{v}^*), \left(1 + \frac{\zeta^*}{2}\right) \boldsymbol{\chi}(\mathbf{v}^*), \left(1 - \frac{\zeta^*}{2}\right) \boldsymbol{\chi}_{d+2}(\mathbf{v}^*)\right\} \tag{B.8}$$

Since Eq. (B.3) is linear, the general solution can be represented as superposition of those for a given wavelength (or wavevector). Consequently, it is convenient to convert the kinetic equation to an algebraic equation by Fourier–Laplace transformation. The transformed perturbation is defined by

$$\delta\tilde{f}^*(\mathbf{k}^*, \mathbf{v}^*, z^*) = \int_0^\infty dt^* \, e^{-z^* t^*} \int d\mathbf{r}^* \, e^{i\mathbf{k}^* \cdot \mathbf{r}^*} \, \delta f^*(\mathbf{r}^*, \mathbf{v}^*, t^*) \tag{B.9}$$

The transformed linear kinetic equation reads

$$\begin{aligned} \delta\tilde{f}^*(\mathbf{k}^*, \mathbf{v}^*, z^*) = (z^* + 1 - i\mathbf{k}^* \cdot \mathbf{v}^*)^{-1} \Bigg[& \delta f^*(\mathbf{k}^*, \mathbf{v}^*, t^* = 0) \\ & + f_H^*(\mathbf{v}^*) \sum_i \tilde{C}_i(\mathbf{k}^*, z^* \mid \delta f^*) \, \psi_i(\mathbf{v}^*) \Bigg] \end{aligned} \tag{B.10}$$

This formal solution is completed by using it in Eq. (B.5) to obtain closed expressions for the $\tilde{C}_i$,

$$\tilde{C}_i(\mathbf{k}^*, z^* \mid \delta f^*) = \sum_j \left[\mathsf{I} - \mathsf{M}(\mathbf{k}^*, z^*)\right]_{ij}^{-1} \tilde{C}_{0j}(\mathbf{k}^*, z^* \mid \delta f^*) \tag{B.11}$$

where I is the unit matrix of dimension $d+2$,

$$\tilde{C}_{0i}(\mathbf{k}^*, z^* \mid \delta f^*) = \int d\mathbf{v}^* \, \chi_i(\mathbf{v}^*)(z^* + 1 - i\mathbf{k}^* \cdot \mathbf{v}^*)^{-1} \, \delta f^*(\mathbf{k}^*, \mathbf{v}^*, t^* = 0) \tag{B.12}$$

$$M_{ij}(\mathbf{k}^*, z^*) = \int d\mathbf{v}^* \, f_H^*(\mathbf{v}^*) \, \chi_i(\mathbf{v}^*) \, \psi_j(\mathbf{v}^*)(z^* + 1 - i\mathbf{k}^* \cdot \mathbf{v}^*)^{-1} \tag{B.13}$$

The possible excitations of the gas are determined from the singularities of the general solution given by Eq. (B.10) in the complex z-plane. These include the single particle excitations damped at the collision frequency, given by $z^* = -1 + i\mathbf{k}^* \cdot \mathbf{v}^*$. They represent free particle motion modulated by pure exponential damping. The additional excitations arise from the solutions to $\det[\mathsf{I} - \mathsf{M}(\mathbf{k}^*, z^*)] = 0$. As described below, these solutions are the $d+2$ hydrodynamic modes. Finally, $\mathsf{M}(\mathbf{k}^*, z^*)$ has a branch point at $z^* = -1$ that leads to the non-hydrodynamic, or kinetic, modes.

The simplest illustration of these excitations occurs for the special case of initial transverse velocity perturbations. The components of the flow field $\mathbf{u}$ can be chosen to be the longitudinal component, $\widehat{\mathbf{k}} \cdot \mathbf{u}$, where $\widehat{\mathbf{k}}$ is a unit vector along $\mathbf{k}$, and $d-1$ transverse components $\{\hat{\mathbf{e}}_i \cdot \mathbf{u}\}$, where the $\hat{\mathbf{e}}_i$, $i = 1, \ldots, d-1$, are unit vectors orthogonal to $\mathbf{k}$. Consider an initial perturbation of the form

$$\delta f^*(\mathbf{k}^*, \mathbf{v}^*, t^* = 0) = f_H^*(\mathbf{v}^*)\, \hat{\mathbf{e}}_1 \cdot \mathbf{v}^* \tag{B.14}$$

corresponding to a perturbation of the transverse velocity flow field, $\delta \mathbf{u}(\mathbf{k}, 0) = [v_H(0)/2]\, \hat{\mathbf{e}}_1$. Then, all $\widetilde{C}_i$ vanish except the one associated with the component of the velocity $v_1^* = \hat{\mathbf{e}}_1 \cdot \mathbf{v}^*$, and from Eqs. (B.10)–(B.12) it is obtained

$$\begin{aligned} \delta\tilde{f}^*(\mathbf{k}^*, \mathbf{v}^*, z^*) &= (z^* + 1 - i\mathbf{k}^* \cdot \mathbf{v}^*)^{-1} f_H^*(\mathbf{v}^*)\, \hat{\mathbf{e}}_1 \cdot \mathbf{v}^* \\ &\quad \times \left[1 + \sqrt{2} \left(1 + \frac{\zeta^*}{2} \right) \hat{\mathbf{e}}_1 \cdot \widetilde{\mathbf{C}}(\mathbf{k}^*, z^*) \right] \end{aligned} \tag{B.15}$$

with

$$\hat{\mathbf{e}}_1 \cdot \widetilde{\mathbf{C}}(\mathbf{k}^*, z^*) = \frac{i}{\sqrt{2}\, k^*} \Phi\left(\frac{z^* + 1}{ik^*} \right) \left[1 - \frac{i}{k^*} \left(1 + \frac{\zeta^*}{2} \right) \Phi\left(\frac{z^* + 1}{ik^*} \right) \right]^{-1} \tag{B.16}$$

where $\Phi(x)$ is the complex probability integral

$$\Phi(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dv\, e^{-v^2} (v - x)^{-1} \tag{B.17}$$

In addition to the damped single particle excitations mentioned above, the dynamical response to transverse perturbations arises from the spectrum of $\hat{\mathbf{e}}_1 \cdot \widetilde{\mathbf{C}}(\mathbf{k}^*, z^* \mid \delta f^*)$ as well, i.e., the branch cut of $\Phi[(z^* + 1)/ik^*]$ and the solutions to

$$\frac{i}{k^*} \left(1 + \frac{\zeta^*}{2} \right) \Phi\left(\frac{z^* + 1}{ik^*} \right) = 1 \tag{B.18}$$

This can be rewritten in the same form as the corresponding dispersion relation for transverse excitations in an elastic fluid by a change of variables

$$k^* = \bar{k}\left(1+\frac{\zeta^*}{2}\right) \tag{B.19}$$

$$z^*+1 = (\bar{z}+1)\left(1+\frac{\zeta^*}{2}\right) \tag{B.20}$$

to get

$$\frac{i}{\bar{k}}\,\Phi\left(\frac{\bar{z}+1}{i\bar{k}}\right) = 1 \tag{B.21}$$

There is only one solution to this equation, $\bar{z} = -z_s(\bar{k})$, representing the $(d-1)$-fold degenerate hydrodynamic shear mode ($z_s(\bar{k})$ vanishes as $\bar{k} \to 0$). The function $z_s(\bar{k})$ is precisely the same as that from the kinetic model for elastic collisions except with $k^* \to \bar{k}$. It is a real, positive, monotonically increasing function defined for all $\bar{k} < \sqrt{\pi}$. In terms of z^* and k^* this hydrodynamic mode is

$$z^* = \frac{\zeta^*}{2} - \left(1+\frac{\zeta^*}{2}\right) z_s\left(\frac{k^*}{1+\frac{1}{2}\zeta^*}\right) \tag{B.22}$$

This expression is defined for $k^* < (1+\frac{1}{2}\zeta^*)\sqrt{\pi}$. For small k^* the Navier–Stokes approximation is recovered:

$$z^* \simeq \frac{\zeta^*}{2} - \frac{k^{*2}}{2+\zeta^*} \tag{B.23}$$

Note that z^* does not vanish as $k^* \to 0$ since the velocity perturbation is reduced by $T_H^{1/2}$ and the cooling generates an exponential growth at half the cooling rate in these dimensionless variables. The dimensionless shear viscosity η^* can be identified from the coefficient of k^{*2} by writing the Navier–Stokes equations in the same variables,

$$\eta^* \equiv \frac{\eta \nu_H}{p_H} = \left(1+\frac{\zeta^*}{2}\right)^{-1} \tag{B.24}$$

where p_H is the pressure. This result can be used to imbed the shear viscosity for the Boltzmann equation for inelastic collisions in the kinetic model by a suitable choice of ν_H. Let η_0 denote the Boltzmann shear viscosity in the elastic limit and define a characteristic frequency by $\nu_0 \equiv p_H/\eta_0$. Then Eq. (B.24) can be written

$$\frac{\eta}{\eta_0} = \frac{\nu_0}{\nu_H + \frac{1}{2}\zeta_H} \tag{B.25}$$

This viscosity relative to its elastic limit has been calculated recently from the Boltzmann equation for $d = 3$.[8] The form (B.25) can be made to agree with the Boltzmann result with the choice

$$\nu_H = \nu_0[1 - \tfrac{1}{4}(1-\alpha)^2] - \zeta_H \tag{B.26}$$

Finally, it is instructive to invert the Laplace transform in this example of transverse perturbations to show the approach of the general solution for $\hat{\mathbf{e}}_1 \cdot \delta\mathbf{u}(\mathbf{k}, t)$ to that from hydrodynamics for long times. Using that $\mathbf{C} = \sqrt{2}\, \delta\mathbf{u}/v_H(t)$, the inverse Laplace transform of Eq. (B.16) yields

$$\frac{\hat{\mathbf{e}}_1 \cdot \delta\mathbf{u}(\mathbf{k}, t)}{\hat{\mathbf{e}}_1 \cdot \delta\mathbf{u}(\mathbf{k}, 0)} = \left(\frac{T_H(t)}{T_H(0)}\right)^{1/2} \frac{1}{2\pi i} \int_c dz^*\, e^{z^* t^*} \frac{\frac{i}{k^*} \Phi\left(\frac{z^*+1}{ik^*}\right)}{1 - \frac{i}{k^*}\left(1 + \frac{\zeta^*}{2}\right) \Phi\left(\frac{z^*+1}{ik^*}\right)} \tag{B.27}$$

The integrand in the above expression is analytic for $\operatorname{Re} z^* > \zeta^*/2$ so the Bromwich integral can be taken to extend from $-i\infty + \frac{1}{2}\zeta^*$ to $i\infty + \frac{1}{2}\zeta^*$. Next, using the change of variables in Eqs. (B.19) and (B.20) this simplifies to

$$\frac{\hat{\mathbf{e}}_1 \cdot \delta\mathbf{u}(\mathbf{k}, t)}{\hat{\mathbf{e}}_1 \cdot \delta\mathbf{u}(\mathbf{k}, 0)} = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} d\bar{z}\, e^{\bar{z}\bar{t}} \frac{\frac{i}{\bar{k}} \Phi\left(\frac{\bar{z}+1}{i\bar{k}}\right)}{1 - \frac{i}{\bar{k}} \Phi\left(\frac{\bar{z}+1}{i\bar{k}}\right)} \tag{B.28}$$

with $\bar{t} \equiv t^*(1 + \frac{1}{2}\zeta^*)$. This is the same result as obtained from the BGK kinetic model for elastic fluids, $\hat{\mathbf{e}}_1 \cdot \delta\mathbf{u}_0(\mathbf{k}, t)$, except that k and t are replaced by $\bar{k}$ and $\bar{t}$,

$$\hat{\mathbf{e}}_1 \cdot \delta\mathbf{u}(\mathbf{k}, t) = \hat{\mathbf{e}}_1 \cdot \delta\mathbf{u}_0\left(\frac{\mathbf{k}^*}{1 + \zeta^*/2}, t^*\left(1 + \frac{\zeta^*}{2}\right)\right) \tag{B.29}$$

The kinetic model therefore provides a precise mapping of the macroscopic transverse velocity field for granular fluids to that for normal fluids. In particular, it shows that the known relaxation of kinetic modes to a long time, long wavelength hydrodynamic description is the same up to considerations of scaling. To be more explicit, the integral in (B.28) can be evaluated as part of a closed contour enclosing the branch cut of Φ and the hydrodynamic pole of (B.21):

$$\frac{\hat{\mathbf{e}}_1 \cdot \delta \mathbf{u}(\mathbf{k}, t)}{\hat{\mathbf{e}}_1 \cdot \delta \mathbf{u}(\mathbf{k}, 0)} = A(\bar{\mathbf{k}})\, e^{-z_s(\bar{k})\,\bar{t}} + e^{-\bar{t}} B(\mathbf{k}, \bar{t}) \tag{B.30}$$

where

$$A(\bar{\mathbf{k}}) = -\left[\frac{d \ln \Phi((\bar{z}+1)/i\bar{k})}{d\bar{z}}\right]^{-1}_{\bar{z}=-z_s} \tag{B.31}$$

$$B(\mathbf{k}, \bar{t}) = \operatorname{Re} \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\, e^{i\omega\bar{t}}\, \frac{i}{\bar{k}}\, \Phi\left(\frac{\omega^-}{\bar{k}}\right)\left[1 - \frac{i}{\bar{k}}\, \Phi\left(\frac{\omega^-}{\bar{k}}\right)\right]^{-1} \tag{B.32}$$

and $\omega^- \equiv \omega - i\varepsilon$, with the limit $\varepsilon \to 0^+$ implicit. These results are still exact. They show that the transverse velocity field perturbations have two types of dynamical response. One is hydrodynamic with relaxation on a hydrodynamic time scale $\bar{t}_s = z_s^{-1}(\bar{k})$ which diverges as $k \to 0$. The second is a fast kinetic relaxation on the time scale $\bar{t}_m = 1$. Thus for long times and long wavelength perturbations the hydrodynamic relaxation dominates and a simpler hydrodynamic description is justified.

The analysis is similar for the response to more general initial perturbations. Although there is not an exact scaling relation between the elastic and inelastic cases, as in (B.29), it still follows that there is a separation into short time kinetic and long wavelength hydrodynamic excitations. The hydrodynamic excitations include the above $d-1$ shear modes, plus three additional modes. For elastic collisions the latter are the two sound modes and the heat diffusion mode. In the case of inelastic collisions these three modes are more complex and do not necessarily have this same physical interpretation. The hydrodynamic description will be presented in detail elsewhere.

APPENDIX C. DETAILS OF RET KINETIC MODEL

In this appendix the steps leading to Eq. (5.10) are outlined. The analysis follows that for the elastic case.[11] The first term on the right side of (5.9) is calculated directly:

$$\mathscr{P} J'_E[f] = \mathscr{P} J_E[f] - \mathscr{P}\, \frac{\zeta}{2} \frac{\partial}{\partial \mathbf{v}} \cdot (\mathbf{V} f) \tag{C.1}$$

The first term gives contributions from the collisional transfer parts of the momentum and energy fluxes, $\mathsf{P}^c[\mathbf{r},\mathbf{v}\mid f]$ and $\mathbf{q}^c[\mathbf{r},\mathbf{v}\mid f]$, just as in the elastic case (although the functional forms differ, as indicated below). In addition, the projection of $J_E[f]$ onto the kinetic energy gives rise to the cooling term on the right side of (4.4). This is canceled exactly by the entire contribution from the second term of (C.1), leaving

$$\begin{aligned}\mathscr{P}J'_E[f]=-\frac{\beta}{n}f_{\ell H}\Big[&\mathbf{V}\nabla:\mathsf{P}^c[\mathbf{r},\mathbf{v}\mid f]\\&+\frac{d}{2c}\left(\frac{m\beta}{d}V^2-1\right)(\nabla\cdot\mathbf{q}^c[\mathbf{r},\mathbf{v}\mid f]+\mathsf{P}^c[\mathbf{r},\mathbf{v}\mid f]:\nabla\mathbf{u})\Big]\end{aligned}\qquad\text{(C.2)}$$

with

$$\begin{aligned}P^c_{ij}=\frac{1+\alpha}{4}m\sigma^d\int d\mathbf{v}_1\int d\mathbf{v}_2\int d\hat{\boldsymbol{\sigma}}\,\Theta(\mathbf{g}\cdot\hat{\boldsymbol{\sigma}})(\mathbf{g}\cdot\hat{\boldsymbol{\sigma}})^2\,\hat{\sigma}_i\hat{\sigma}_j\\\times\int_0^1 d\lambda\,\chi[\mathbf{r}-(1-\lambda)\,\boldsymbol{\sigma},\mathbf{r}+\lambda\boldsymbol{\sigma}\mid n]\,f(\mathbf{r}-(1-\lambda)\,\boldsymbol{\sigma},\mathbf{v}_1,t)\,f(\mathbf{r}+\lambda\boldsymbol{\sigma},\mathbf{v}_2,t)\end{aligned}\qquad\text{(C.3)}$$

$$\begin{aligned}\mathbf{q}^c=\frac{1+\alpha}{4}m\sigma^d\int d\mathbf{v}_1\int d\mathbf{v}_2\int d\hat{\boldsymbol{\sigma}}\,\Theta(\mathbf{g}\cdot\hat{\boldsymbol{\sigma}})(\mathbf{g}\cdot\hat{\boldsymbol{\sigma}})^2\,(\mathbf{G}'\cdot\hat{\boldsymbol{\sigma}})\,\hat{\boldsymbol{\sigma}}\\\times\int_0^1 d\lambda\,\chi[\mathbf{r}-(1-\lambda)\,\boldsymbol{\sigma},\mathbf{r}+\lambda\boldsymbol{\sigma}\mid n]\,f(\mathbf{r}-(1-\lambda)\,\boldsymbol{\sigma},\mathbf{v}_1,t)\,f(\mathbf{r}+\lambda\boldsymbol{\sigma},\mathbf{v}_2,t)\end{aligned}\qquad\text{(C.4)}$$

Here $\mathbf{G}'=\frac{1}{2}(\mathbf{v}_1+\mathbf{v})-\mathbf{u}(\mathbf{r},t)$. The collisional transfer contributions vanish at low density, but dominate at high densities. Similarly, the cooling rate ζ for the RET is found to be

$$\begin{aligned}\zeta=(1-\alpha^2)\frac{m\sigma^{d-1}}{4dnk_{\mathrm{B}}T}\int d\mathbf{v}_1\int d\mathbf{v}_2\int d\hat{\boldsymbol{\sigma}}\,\Theta(\mathbf{g}\cdot\hat{\boldsymbol{\sigma}})(\mathbf{g}\cdot\hat{\boldsymbol{\sigma}})^3\\\times\chi[\mathbf{r},\mathbf{r}+\boldsymbol{\sigma}\mid n]\,f(\mathbf{r},\mathbf{v}_1,t)\,f(\mathbf{r}+\boldsymbol{\sigma},\mathbf{v}_2,t)\end{aligned}\qquad\text{(C.5)}$$

The second term on the right side of (5.9) is replaced by the approximation

$$(1-\mathscr{P})\,J'_E[f]\to(1-\mathscr{P})\,\lambda f\qquad\text{(C.6)}$$

The free parameter is divided into an average collision frequency plus a velocity dependent perturbation,

$$\lambda[\mathbf{v}\mid f]=\nu[f]+\delta\nu[\mathbf{v}\mid f]\qquad\text{(C.7)}$$

where ν is independent of the velocity. To suggest the choice for $\delta\nu[\mathbf{v} \mid f]$, consider the model collision operator evaluated at the *local* HCS state, which can be written as

$$J'_E[f_{\ell H}] \to J'_E[f_{\ell H}] - (1-\mathscr{P})(\delta\nu(\mathbf{v})\, f_{\ell H} + J'_E[f_{\ell H}]) \tag{C.8}$$

Use has been made of the property $(1-\mathscr{P})\,\nu f_{\ell H} = \nu(1-\mathscr{P})\, f_{\ell H} = 0$. For consistency in (C.8) we choose

$$\delta\nu(\mathbf{v}) = -f^{-1} J'_E[f_{\ell H}] \tag{C.9}$$

The interpretation of λf is the change of f due to collisions that affect only the component $(1-\mathscr{P})\, f$. There are two parts. The first represents an average collision rate ν depending on the local temperature and density. The second represents an additional collision rate due to spatial inhomogeneities of the local HCS state (recall $J'_E[f_H]$ vanishes for the *uniform* HCS). The latter is a collisional transfer effect associated with the difference in position of the colliding particles.

The inclusion of $\delta\nu(\mathbf{v})$ leads to a quantitative improvement of the predicted transport coefficients. However, the price is an additional complex velocity dependence, beyond the simple polynomial dependence of $\mathscr{P}J'_E[f]$. This undesired complication of the kinetic model can be eliminated by retaining only the lowest order polynomial dependence of $(1-\mathscr{P})\,\delta\nu f$. Closer inspection shows that the Chapman–Enskog solution (to first order in the gradients) and transport coefficients depend only on moments of $J'_E[f_{\ell H}]$ with respect to two functions in the subspace of $(1-\mathscr{P})$, namely $\mathsf{D}(\mathbf{V})$ and $\mathbf{S}(\mathbf{V})$ defined in Eq. (5.12). Retaining only the projections of $\delta\nu f$ along these two functions gives

$$\lambda \to \nu - (f_{\ell H}/f)[\mathsf{A} : \mathsf{D}(\mathbf{V}) + \mathbf{B}\cdot\mathbf{S}(\mathbf{V})] \tag{C.10}$$

where

$$\mathsf{A} = \frac{\int d\mathbf{V}\, \mathsf{D} J_E[f_{\ell H}]}{\int d\mathbf{V}\, f_{\ell H} \mathsf{D} : \mathsf{D}}, \qquad \mathbf{B} = \frac{\int d\mathbf{V} \mathbf{S}\, J_E[f_{\ell H}]}{\int d\mathbf{V}\, f_{\ell H} S^2} \tag{C.11}$$

The resulting kinetic equation is given by Eqs. (5.10) and (5.11).

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